AN INTRODUCTION TO THE THEORY OF RELATIVITY

L. BOLTON

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BY

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μηδεὶς ἀγεωμέτρητος εισίτω.—PLATO (alleged)

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PREFACE

THE Theory of Relativity may very well prove to be the most important single contribution yet made to intellectual thought. If the theory is true it means nothing less than that physical science has at length broken through the crust of the phenomenal and apparent. The mechanism of nature is to be sought in something as yet conceivable only mathematically.

It is not to be expected that a theory of this novelty and scope can be other than difficult. No one can be surprised if he finds the general drift hard to grasp. This was indeed by far the most serious difficulty encountered by the writer. It is nothing but literal fact that he found it a greater obstacle to a general understanding of the subject than the details of the advanced mathematical work. Try as he would the drift eluded him. The main, almost the sole, object of the present book is to meet this difficulty, all other considerations being subordinate. He has written the book with a very lively recollection of his own troubles,

and he hopes that it may be of service to others in like case.

The great question is, What is it all about? To this question some give one answer and some another; but none, to the writer's knowledge, give so clear an answer as Einstein himself,* and even he answers it by implication rather than directly. Still the implication of his definitions of the Special, or Restricted, and of the General principles is so plain that there is no mistaking it. His definition of the Restricted Principle, which need not be given here, as it is fully dealt with in the following pages, is a compendium of the special theory and it is easily generalized. His definition of the General Principle simply repeats the definition of the Restricted Principle in wider terms, and he makes it quite clear that Relativity as a whole is *the theory of the statement of general physical laws in forms common to all observers.* It is something of a puzzle why other writers of authority have not given this fact a more prominent place and stated it plainly and explicitly. It may have been because it seemed so obvious as not to require emphasis, but to the writer's mind the greater part of the mystery which has surrounded the subject has arisen through failure to grasp it. It was certainly so in his own case.

* "Relativity, the Special and the General Theory." By Albert Einstein, Ph.D. Translated by R. W. Lawson. Fifth Edition. Crown 8vo, 5s. net. (Methuen & Co., Ltd.)

When he realized it, the whole subject, till then a hopeless jig-saw puzzle, seemed to arrange itself of its own accord. The "Scientific American," in their remarks on the award of the Eugene Higgins Prize,* which the writer was fortunate enough to win, were good enough to compliment him on the "extraordinarily fine judgment which he used in deciding just what he would say and what he would leave unsaid". As a matter of fact, what he did was to say what was strictly relevant to this main issue and side-track what was not. He hopes that whatever the shortcomings of the present book may be, he has at least left the reader's mind clear on this all-important point.

The writer, therefore, has followed Einstein in this general conception of the subject; though the treatment differs very considerably in detail. (The book is, to a large extent, the winning essay extended to twelve or thirteen times its length.) The object is to show that the conclusions of the subject develop easily and naturally out of the search for a general mode of statement of physical laws. All matter which is not strictly relevant to this end is either omitted altogether, or where the amount of public attention which has been directed to certain points forbids their exclusion, it is expressly stated that

* An account of this contest together with a selection of the essays and other matter is being issued under the title "Relativity and Gravitation". Crown 8vo, 7s. 6d. net. (Methuen & Co., Ltd.)

the discussion is a digression (see Chapter X). No attempt at exhaustive, or indeed wide treatment, has been made. Necessarily, in order to say what Relativity is about, considerable detail is required, but nothing has been introduced beyond what is absolutely necessary to this end.

The present book departs from the essay in one important respect. Mathematical symbols are used with considerable freedom. The writer contrived to avoid them in the essay; but, while writing it, he was conscious all the time that he was thinking mathematics, and that his exposition, such as it was, suffered by the absence of symbols. It is impossible to avoid mathematics, and the motto on the title-page is meant to imply this fact. It is the notice which Plato is alleged to have put up warning indifferent mathematicians off his premises. Physical laws must be stated in mathematical terms to be of any value, and the subject is therefore essentially mathematical. To expect a non-mathematical treatment of Relativity is as reasonable as to expect a non-mathematical treatment of the Integral Calculus. At the same time, a very small amount of mathematical knowledge indeed is required for a general grasp of the subject. The mathematical knowledge assumed in this book is exiguously small. Einstein says that his book presumes a standard of education corresponding to that of a university matriculation examination.

The present book, the writer thinks, requires less, nothing in fact beyond simple equations and Euclid I, 47 (the Theorem of Pythagoras). Wherever a proof is given it is written out in great detail, and this may at first sight give the impression of overmuch mathematics. This extreme detail may be unnecessary, but the writer felt that it was better to be on the safe side.

Perhaps the most serious difficulty after that of understanding the drift of the subject is the necessity for getting rid of all metaphysical notions. Philosophic questions may be considered at the end of the subject, but at the beginning and in the course of the subject they are out of place and misleading. The difficulty of suppressing metaphysical considerations is of a peculiarly insidious kind, and it requires a distinct mental effort to overcome it. Particular attention should therefore be paid to what is said in the text on this point.

Next to this is the necessity for understanding the nature of reference frames and systems, and their relation to an observer's point of view. It is hoped that Chapter III will clear up this important matter.

There is one matter of detail in which the prize essay has been departed from, and that is the treatment of rotation (chapter XIII). In the essay the writer borrowed his illustration respecting

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measured times and lengths on a rotating system from Einstein, and he still thinks that Einstein's illustration is most apt and telling when properly understood. Unfortunately, experience has proved to him that it raises so many irrelevant suggestions as to make it practically useless. He has therefore most reluctantly abandoned it.

Naturally, a very large number of books and other publications have been laid under contribution, and the writer gratefully admits his obligation to the authors. He has endeavoured to do full justice in the way of acknowledgment; but if there are any omissions he hopes the authors will realize how impossible it is to acknowledge every detail.

In conclusion, the writer desires to thank the friends who have helped him by their criticisms and suggestions. Their help has been invaluable to him. He also desires to thank Mr. F. E. Smith of Bedford School for drawing the diagrams.

L. B.

BEDFORD, 23 *May*, 1921.

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AN INTRODUCTION TO THE THEORY OF RELATIVITY

CHAPTER I

INTRODUCTORY

IF the reader expects a non-mathematical treatment of the Theory of Relativity, it is to be feared that he will be disappointed. The subject matter belongs to mathematical physics; indeed, in a sense, it includes the whole of mathematical physics, for it deals with the mathematical expression of those descriptive statements of fact which are called physical laws. It is, therefore, impossible to avoid mathematics; we must think more or less in mathematical terms, even though no symbols are actually written down. Fortunately, the amount required for a general understanding of the subject is very small, and should present no difficulty, with the full explanations which it is proposed to give. The real obstacle in the way of a generally intelligible treatment is not so much the fact that mathematics is necessarily involved as the unfamiliar character of the subject matter, which renders it somewhat difficult to give at the start a clear indication of its drift. It is not difficult to devise a form of words which substantially covers the ground, but the words

themselves gather their meaning from the subject, and are interpreted differently according to the reader's knowledge of it. The observations immediately following, therefore, must not be taken as exhaustive, but it is thought that they will give a reasonably clear preliminary idea of the kind of subject matter with which we shall have to deal.

Phenomena look different to different people, though the phenomenon or thing is physically the same. It is unthinkable that an observer can change the nature of anything by merely looking at it. Now, Relativity seeks to reconcile these differences and to determine statements of fact which shall be independent of different observers; which shall describe phenomena independently of any particular point of view. Relativity is the theory of the expression of general physical facts in a way which shall be common to all observers and independent of anyone in particular. Looked at in this way, "Relativity" is not altogether a satisfactory name. It concentrates attention too much on individual points of view, whereas the real object is their elimination. The point is not without importance. The Relativity of Knowledge is a well-known philosophical doctrine, and the name Relativity misleads some persons, more especially if they have an acquaintance with metaphysics, into the belief that Einstein's theory is nothing more than a reassertion of the doctrine in a slightly modified shape.* It is

* Since this paragraph was written, Lord Haldane's book, "The Reign of Relativity," has appeared. This work deals comprehensively with the Relativity of Knowledge. Amongst other things the position of Einstein's theory—more especially those

probably too late now to choose another name, but it is as well to remember that the name Relativity savours of the *lucus a non lucendo* principle.

It is not the writer's purpose to anticipate subsequent discussions by enlarging upon previous attempts at the statements of general physical laws, upon the limitations of these attempts, or upon the suppositions which underlie them. These will appear in due course, but meanwhile as an example it may be stated that until the relativists interfered it had always been thought that in specifying objects or phenomena, measurements of space were to be treated as entirely distinct and independent from those of time, and that general physical laws, that is to say, general statements of fact independent of particular observers, could be framed on that basis. The following instance illustrates the relativist position in regard to this important matter, but the points raised will be dealt with in greater detail later on.

Let us take some simple physical object such as a cube, which we shall suppose to be opaque and to have its edges less than the distance between the pupils of the observer's eyes. The qualifications are of no great importance, but they simplify the discussion somewhat. If now we take our station so that our eyes are as nearly as possible opposite the middle of one of the faces, what we see is simply a square as in the first diagram of Fig. 1. As we move round to the left, a second face comes into view, the top and bottom sides of the two faces losing their apparent parallelism, and tending to

parts of it relating to space and time measurements—in relation to this doctrine is discussed.

vanishing points according to the rules of perspective. We do not get quite the same impression from both eyes, since they are at different station points, and this, together with differences in illumination, the presence of intervening objects, and, above all, previous experience, gives us a sense of relief or solidity depending upon our distance away from the cube. As we move still further round and occupy other points of view, we get the impressions shown by the other diagrams of

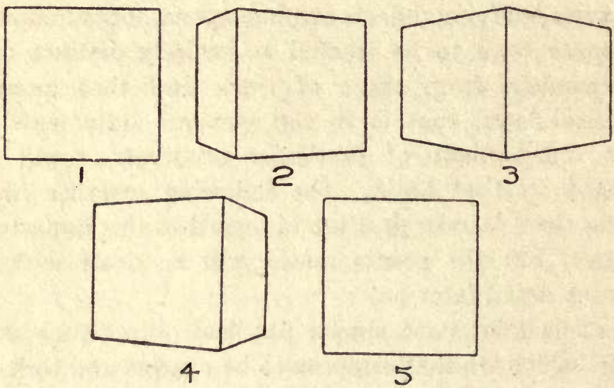

Fig. 1.

the figure, until when we come opposite the next face of the cube, we see a square as at first ; and so on. We get yet another set of impressions by first taking our stand in a position corresponding to diagram No. 3 of Fig. 1, and then moving upwards so that our eyes, when in the next position, are level with the top of the cube. The succession of impressions as we move round over the top is shown in Fig. 2, ending up with a square set diamond-wise, which we see when we are

looking down on the cube from some point over the middle of the top. We never see the whole cube at once; we can, in fact, never see more than three faces at the same time, but by combining the impressions got by occupying a number of station points, we are able to form an opinion as to its shape. As the physicist would say, we construct a theory of its shape. We say that it is a solid figure, square in plan, front eleva-

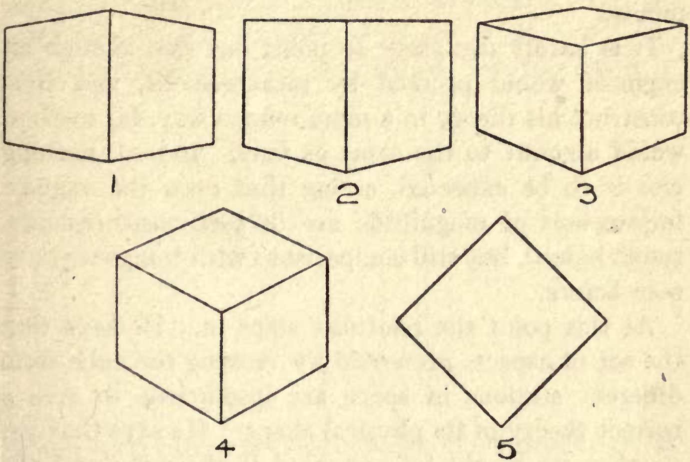

Fig. 2.

tion and side elevation. We could have made the cube go through another series of aspects by approaching it or moving further away; and, of course, we could have made it present itself in all its aspects merely by turning it round, and moving it nearer or further off. These modifications are, however, immaterial. The point is that we get an idea of what the cube is as a physical object by looking at it in various aspects from different view points and collating the results.

Now, in obtaining these aspects we have moved right or left, up or down, and backwards or forwards. All our view points have been arrived at by moving in one or more of these three mutually perpendicular directions. We have, in fact, constructed our theory of physical shape on the supposition that we are dealing with a three-dimensional object in three-dimensional space, and no other considerations have entered our minds.

It is hardly necessary to point out that though an engineer would proceed by measurement, and thus construct his theory in a more refined way, his method would amount to the same as ours. Indeed, nothing else is to be expected, seeing that even the vaguest impressions of magnitude are in fact measurements, rough indeed, but still comparisons with things we have seen before.

At this point the relativist steps in. He says that the set of aspects presented by viewing the cube from different stations in space are insufficient to give a correct theory of its physical shape. He says that yet another set would be presented if the cube and the observer were in relative motion. If the observer were to keep on the move while collecting his impressions, or making his measurements, if this were possible, these would be different from the impressions or measurements he would get while standing still. For example, if during the first observation, when the observer saw the cube as a square, he had been moving through his point of observation sideways, parallel with the face, he would have observed the figure, not as a square, but as a parallelogram (Fig. 3), having its width less

than its height. In fact, all the different figures would have been crushed up, like an accordion, in the direction of motion. It is true, says the relativist, that this crushing is imperceptible, but this, so he says, is because the observer's motion is comparatively slow. If it were of the order of the velocity of light, the change of shape would be manifest, and if it were usual for things to move about at such a rate, these changes of shape would be accepted as matters of course. In fact, electrons, which sometimes move with very high velocities, though their magnitudes are too small for

FIG. 3.

direct observation of their changes of shape, do actually exhibit peculiarities of motion which can be shown to be the direct consequences of these changes.

The relativist goes on to say that his statements, so far from causing surprise or incredulity, are only what we ought to expect. By introducing velocity into our theory of shape, we have only done what nature always does, and brought in the element of time. Time and space are never separated in nature, and we have no right to separate them in our theories which are supposed to represent nature. Things exist both in space

and time, and the two are inseparably joined.* Any physical theory of the shape of a thing must therefore include time. It is wrong to consider a thing as existing in three-dimensional space on the one hand, and enduring independently in one-dimensional time on the other. He sometimes puzzles us by saying that an object must be regarded as a four-dimensional thing existing in a four-dimensional continuum, as he calls this combination of time and space, but all he means is that we must take into account in a certain special manner a fourth element, namely, time, in addition to length, breadth, and height, or thickness. If we ask the relativist whether he thinks that space and time are the same, he says, "What I said was that they are inseparable, not that you should not distinguish them. If you like to go into the subject further you will find that your mathematical processes will distinguish them for you quite sufficiently for your purposes." He tells us that he, no more than we, can picture four-dimensional objects, and, furthermore, that he does not want to do it, and that it would not help him much for physical purposes if he could, seeing that he can get all he requires merely by supposing things to be determined by four independent quantities instead of three. But of this more hereafter. In subsequent chapters we shall see what evidence the relativist can produce in support of his strange theories.

Meanwhile there are some preliminary matters requiring attention. A number of words have been used whose meaning is probably not at all clear to the

* "Einstein's Theories of Relativity and Gravitation" (Scientific American Publishing Co., New York; also Methuen & Co., Ltd., London), p. 186.

reader. We have spoken of physics, metaphysics, physical laws, mathematics, space, time, points of view, and so forth; all of which require definition. It is very unlikely, for example, that the reader knows what is meant by a point of view, which is probably the most important for our purposes of any. In the next four chapters we shall deal with these matters and some others required for the subsequent work.

Summary.—Relativity treats of the mathematical expression of general physical laws. For a general understanding of the subject, mathematical ideas are required, but no great proficiency. The difficulty of the theory resides in its novelty. Space and time are inseparable; for example, no physical theory of shape can be framed which excludes time.

CHAPTER II

METAPHYSICS, PHYSICS, AND MATHEMATICS

I. METAPHYSICS

THE "New English Dictionary" defines metaphysics as "that branch of speculative inquiry which treats of the first principles of things, including such concepts as being, substance, essence, time, space, cause, identity, etc."; "theoretical philosophy as the ultimate science of Being and Knowing". The name seems to have referred originally merely to the order in which the books dealing with these subjects occurred in the received edition of Aristotle's writings. These books came after those on physics. By a misinterpretation the preposition μετὰ acquired the meaning of *beyond* or *transcending*, which now attaches to it in this connexion.

Now, seeing that these inquiries go to the very root and essence of things, it would seem only proper and logical to take their results as the foundation of all other knowledge. Unfortunately, though they have engaged the attention of many of the greatest minds from the earliest ages, and are still pursued, no definite conclusions have been reached. The inner nature of space, time, cause, and such like concepts still remains undefined, and those who wish to pursue other branches

of knowledge must therefore find some starting point other than metaphysics.

2. Physics

The physicist accordingly defines things as they present themselves to his observation. We are mainly concerned here with the fundamental concepts, space and time, so we will take these as examples. The physicist does not know what space is; the metaphysician has not told him. But he defines a length, or the distance between two points, on a plane for instance, to be the number of times a given standard or unit measuring rod will have to be laid down endways in successive adjacent positions along the straight line joining the two points so as to reach from one to the other. If the points are not on a plane, but on a curved surface, such as that of the earth, he cannot proceed far in a straight line, and if necessary he modifies this definition in an obvious way. If his measuring rod is what he calls a yard, and he has to lay down the rod so many times, he says the distance between the points, or the length of the interval between them, is the same number of yards. He assumes that his standard measure is rigid, that is, that it does not alter its own length capriciously without his knowledge. If it did so alter, his measurements would be at fault unless everything else altered accordingly, in which case he would have no optical means of knowing it, though mechanical means might perhaps be available. So also for time. An interval of time between two events is for his purposes the number of rotations or the fraction of a rotation of the earth,

the number of oscillations of a pendulum, or the number of vibrations of a sodium atom, or the like, which take place between the occurrence of the two events. In measuring time he assumes a quality corresponding to rigidity in his standard lengths; he assumes that his clocks, as we may call his time-pieces, whatever their nature, do not alter their rates without his knowing it.

Time and space in their physical sense are thus *intervals of time* and *measured lengths*. Physical time and space are entirely distinct from the concepts of duration and extension of the metaphysician. They are time and space as disclosed by measurement, or, if the expression be preferred, they are the behaviour of clocks and measuring rods.

This distinction between physical and metaphysical time and space is all-important in the present subject. If it is not clearly understood that these words are used solely in their physical sense unless otherwise stated, most of what follows will sound paradoxical, or even nonsensical. Much of the misunderstanding of the theory of relativity would be avoided if this distinction were kept in mind.

This seemingly arbitrary way of defining things without previously investigating their nature may be thought to be unsound and liable to error. There are two answers to this. In the first place, whatever may be the whole content of the concept of time or of space, measured time or measured length is part of it, so that the physical sense cannot be wrong; at the worst it can only be inadequate. In the next place, whether actually wrong or merely inadequate, experiment gives means for checking it. Sooner or later deduction based

on the definitions will lead to results which experiment will contradict, and thus show the need for amendment. This procedure was not open to the early philosophers, who did not understand the method of experiment. More will be said on this point in a later chapter when we come to deal with the method of hypothesis. It is sufficient for the present to observe that so far the physical definitions have not led to any such contradictions.

It is, of course, not necessary to assume that the physical definitions comprise the whole content of the concepts of space and time. Some hold that they do, but this opinion is repugnant to many minds. For physical purposes the point is immaterial as long as the results are consistent with experiment.

In addition to definitions, the physicist requires certain fundamental principles in the form of postulates or axioms. These can be treated on similar lines to the definitions, and checked by experiment.

3. Mathematics.

The mathematician meets the metaphysical difficulty in a different way. He simply says that it is not his business to inquire whether his definitions and postulates are accurate representations of things or not. As long as they are not self-contradictory and are mutually consistent they satisfy his requirements. Physical truth and mathematical truth are different things. The definitions and postulates of physics have to agree with nature, those of mathematics need only agree with one another. The truth of Euclid would be

unaffected though such things as squares, straight lines, right angles, and the like never existed. Indeed, it is very unlikely that they do exist. The chances are probably millions to one against the existence of an exact square according to Euclid's definition, and it is quite certain that no one is gifted with faculties refined enough to recognize it if it did exist. As a rule, mathematical definitions agree with natural conditions more or less, since they are generally suggested by them, but it is not necessary that they should, and the mathematician, if he is a pure mathematician and not a physicist, is not concerned.

4. Mathematical Physics

It follows that when mathematical processes are applied to physics, the provisional assumption is made that the definitions and postulates of the mathematician are applicable to physical phenomena. If this assumption is incorrect the mathematical deductions disagree with experimental tests. There have been cases in which disagreement has arisen and has led to important discoveries.

One of the best instances of this is Planck's quantum theory of radiation. Matter is perpetually radiating energy in the form of pulses or waves. These waves vary in length, from the almost infinitesimal proportions of the X-rays—and possibly still smaller rays as yet undetected—up to those of wireless rays, the lengths of which may extend to miles and no one knows how much bigger. In between are the rays which manifest themselves as light and heat. If these rays strike other

bodies they are absorbed, reflected or transmitted. All these rays represent so much energy, and they might be made to do mechanical work by proper appliances. If now we consider any closed region from which no radiation can escape and which none can enter, a continual exchange in the form of radiation takes place between the various bodies present, absorbed rays being in turn emitted in the form of other rays, and thus a state of equilibrium is eventually reached in which all the radiations exactly balance, and no further apparent physical change occurs It had been assumed that this emission of energy took place in accordance with electro-magnetic laws which involve complete continuity when applied to the mechanism of radiation and absorption—that is to say, that the mathematical definition of continuity was satisfied within the limits of experimental error, so that the mathematical processes founded upon this definition could be applied to all the circumstances of the case. It was found, however, that this supposition involved the concentration of an infinite amount of energy in the æther, which would thus drain all the energy out of the bodies in the enclosure, and this was contrary to experience. It was, therefore, assumed that energy was *emitted* by small jumps which are integral multiples of minute definite *quanta*, each quantum being proportional to the wave-length of the emitted radiation. This supposition agrees with the facts. It is not found necessary to suppose that the *absorption* takes place otherwise than continuously.

It is very improbable that the agreement between mathematical definitions and axioms, and physical

facts, is ever complete, but if the discrepancy is within the limits of experimental error it is undetectable.

Summary.—Metaphysics deals with the fundamental nature of things, but it has not led to results sufficiently definite for the purposes of physics and mathematics. The physicist takes things as he observes them, trusting to experiment to correct him. The mathematician is not concerned with physical existence, but only with consistency. The application of mathematics to physics involves hypothesis. The relativist concepts of space and time are physical, not metaphysical.

CHAPTER III

REFERENCE SYSTEMS

THIS chapter deals with the implications of the term "point of view".

If we wish to form an accurate idea with the aid of a map of a tract of country at which we are looking, a good way is to lay on the map a piece of thin transparent celluloid ruled with intersecting lines.

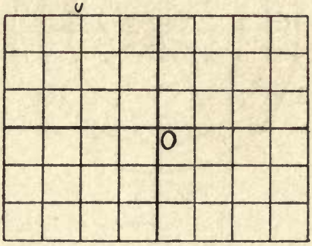

FIG. 4.

The lines, as shown in Fig. 4, should preferably intersect at right angles and divide the celluloid strip into squares, and they should be as numerous as possible, consistent with clearness. We select a pair of these lines, such as the thick ones in the figure, and apply the strip to the map so that their intersection *O* registers with the spot on the map corresponding to our point of view, orienting the scale in any convenient way. If

now we imagine the landscape covered with a network of lines corresponding to those on the scale, and similarly situated, we have a reference frame whereby we may determine the positions of any of the objects in view relative to one another and to ourselves.

The same principle might be applied in other ways. For example, the strip might be replaced by a circular disk, Fig. 5, marked with equidistant concentric circles and lines radiating from their common centre *O*. We might apply this disk to the map as before, and imagine the landscape to be divided up similarly.

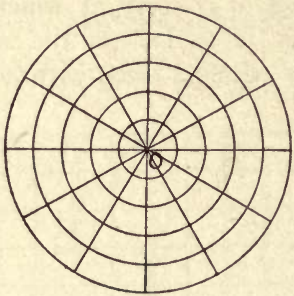

FIG. 5.

It is evident that this procedure can be varied indefinitely according to convenience. Even the equality of the divisions is not absolutely essential; it makes measurements easier, but that is all. If measurements are not required, but only relative positions, irregular lines will often answer our purpose as well as others. The essentials are that some sort of reference frame corresponding to every point of view taken up is required, and that once a reference frame has been selected it must be supposed to remain rigid. If it

alters its configuration it is of no use. A reference frame is the natural correlative of an observer's point of view. The one implies the other.

The physicist also has to determine and record positions. He observes or determines occurrences in what he calls fields of force—that is to say, regions in which various agencies—gravitational, electric, magnetic, mechanical, cohesive, and so on—are acting, together with the effects of these agencies. He proceeds in precisely the same way as the observer in the case of the landscape, and fits out his field with a rigid

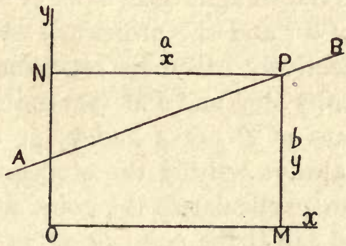

FIG. 6.

reference frame which represents his point of view. The physicist calls his reference frames "co-ordinates" or "systems of co-ordinates".

We proceed to give particulars of the various systems of co-ordinates which will be used in this book.

I. CARTESIAN CO-ORDINATES

The name of this system is derived from the philosopher Descartes, who invented it. The principle is the same as that described in connexion with Fig. 4. We shall first describe the system with reference to a plane, and afterwards extend it to three dimensions.

Referring to Fig. 6, Ox and Oy are two straight lines at right angles, intersecting at a point O, which is called the *origin*. All other points, such as P, are located with reference to these two lines, called the *axes*. Ox is called the *axis of x*, and Oy is the *axis* of y. It does not matter to which of the lines we give these names, but they are usually applied as shown. The position of any point, P, is determined when we know the lengths of the perpendiculars, PM, PN dropped from P upon them, or the lengths of the lines ON, and OM, which are equal to PM and PN each to each. PM or ON is called the *ordinate* of P, and PN or OM is called the *abscissa* of P, and the ordinate and abscissa of a point are collectively called its *co-ordinates*. Thus, if PN is three units long and PM two units, we say that the co-ordinates of P are 3 and 2, or that P is the point (3, 2), always writing the abscissa first. If we do not wish to particularize the point numerically we use letters and call it the point (a, b), or if its position is subject to continual change, as, for instance, when we are considering a point moving along some straight line such as AB, we usually go to the end of the alphabet and call it the point (x, y). Thus we should say, " Let (x, y) be any point P on the straight line AB ".

A system of axes such as has just been described, intersecting at right angles, is called " rectangular axes," but it is sometimes convenient to use " oblique axes " where Ox and Oy do not intersect at right angles, as in Fig. 7. The abscissa PN or OM, and the ordinate PM or ON, are always taken parallel to the axes. The system of nomenclature is the same as before.

It is easy to adapt the Cartesian system to three

dimensions by taking a third axis Oz through the origin, perpendicular to both Ox and Oy, as in Fig. 8, which is a perspective view. The three pairs of axes, Oy

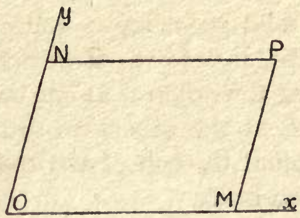

FIG. 7.

and Oz, Oz and Ox, Ox and Oy, now define three planes, like three sides of a box which meet at a corner O, and the axes Ox, Oy, and Oz are the edges of the box,

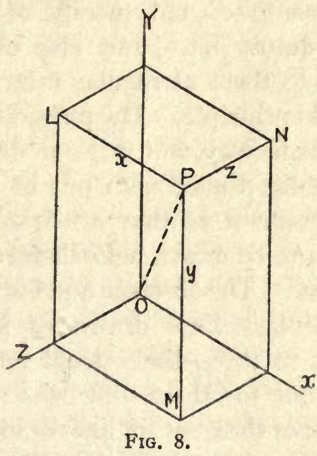

FIG. 8.

meeting at the same corner. The position of any point, P, in space, is determined by its perpendicular distances, PL, PM, and PN from these planes. Con-

formably with the notation used already we may call the point P (a, b, c) or (x, y, z), as the case may be, using any convenient letters. The three distances, PL, PM, and PN are each called *ordinates* (the word abscissa is not used in solid geometry) ; collectively they are called *co-ordinates*, as before. If we complete the box-like figure having the origin O at one end of a diagonal and the point P at the other, we can see that there are four lines equal to each of the co-ordinates of P, like the edges of a brick.

Oblique axes are seldom or never used in solid geometry, and they need not detain us.

There is one slight difference between the reference frame of the physicist and that which we supposed the observer of the landscape to use. The reference frame of the latter consisted not merely of two mutually perpendicular datum lines, but also of a number of others parallel to them at regular intervals. There is no real change in principle. The physicist *could* proceed in exactly the same way, but it is not always necessary. All that is usually needed is to put in any subsidiary lines by measurement as they are wanted, instead of supposing them to be drawn beforehand like the rulings on squared paper. The abscissæ and ordinates of points are these subsidiary lines drawn *ad hoc*. A similar remark applies to three-dimensional reference frames. A reference frame for three dimensions, if completed, would be a mass of three sets of lines of indefinite length, the members of each set being parallel respectively to the three axes, and perpendicular to those of the other two sets.

2. Polar Co-ordinates

These are the same as the second kind of reference frame considered in our imaginary survey of a tract of country. In this case we choose, as in Fig. 9, some convenient datum line, OA, which is called the *initial line*, and a point O in it corresponding to the origin in Cartesian co-ordinates, which is called the *pole*. Any point P is determined by its distance OP from O, and by the angle POA between the lines OP and OA. OP, which is usually denoted by the letter r, is called the *radius vector*, and the angle POA, usually written θ,

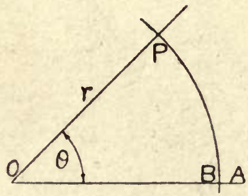

Fig. 9.

the *vectorial angle*. The polar co-ordinates of P are thus r and θ, or P is called the point (r, θ), always writing the radius vector first. When r is given, the point P obviously must lie on a circle PB, having r as radius and centre O. This circle corresponds to one of the circles which we considered in connexion with Fig. 5, though it is never the practice to draw it. OP corresponds to one of the radial lines in Fig. 5. It is obvious that polar co-ordinates are nothing more than the range and bearing of the artilleryman, O being the gun position, P the target, OP, or r, the range, and θ the bearing from the zero line OA.

In polar co-ordinates for three dimensions a second

angle is used. Let P, Fig. 10, be a point in space, O the pole, OA the initial line. Take any fixed line OZ perpendicular to OA. To fix our ideas, suppose that OA is in a horizontal plane, and OZ vertical; but the actual directions are immaterial, as long as the lines are mutually perpendicular. Drop a perpendicular, PM, from P on to the horizontal plane through O, and join OM. Let θ be the angle which OP makes with the vertical OZ, and let ϕ be the angle which OM makes with OA. ϕ is therefore in the horizontal plane and θ

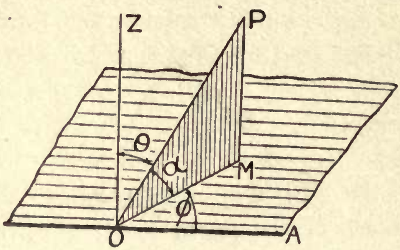

FIG. 10.

in a vertical plane. Let r be the length of the radius vector OP. Then the polar co-ordinates of P are (r, θ, ϕ) written in that order. Polar co-ordinates thus correspond to range, angle of sight, and bearing in gunnery, the difference being that the angle of sight, a, is measured from the horizontal plane, and θ from the vertical line, so that $\theta = 90° - a$. If O be the centre of the earth and P a point on its surface, ϕ is the longitude of P, and a the latitude. It is clear that a system of parallels of latitude and meridians of longitude is a reference system adapted to a sphere.

3. Gaussian Co-ordinates

The straight lines and circles of which the foregoing reference frames are composed are only particular or limiting cases of curves. A perfectly general form of reference frame would therefore consist of sets of curves, as in Fig. 11, which represents such a reference frame for two dimensions of space. The curves may be on any surface, the surface of the earth for example. These systems are called Gaussian co-ordinates, after Gauss, the mathematician, who first used them.

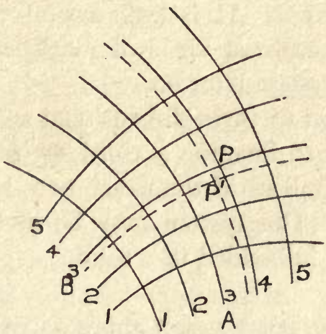

Fig. 11.

Gaussian co-ordinates consist of a set of curves A indefinite in number, drawn according to any regular plan, and crossing a similar set B. All the curves A intersect the curves B, but none of the curves of either set intersect those of the same set. They must be capable of covering the surface on which they are drawn continuously—that is to say, if P and P' are two points near one another, it must be possible to draw separate curves of one or both sets through the points, however close together they may be. A good

idea of this system of co-ordinates may be gathered by imagining it to be a network whose meshes differ but slightly in shape from neighbouring ones, but the difference may mount up considerably when the meshes are far apart. Any point P is located by the intersection of two curves of different sets. Thus, in Fig. 11, P is at the intersection of the curves 4 of set A and 3 of set B. Though the curves are drawn according to some regular plan, successive ones do not generally occur at equal intervals, and the system, therefore, does not lend itself to measurement as with the systems previously considered. It is only useful for embodying the highly generalized ideas of mathematicians. We shall hear of it again later on.

The extension to three-dimensional space is obvious. We have only to imagine a third set of curves intersecting the members of the other two, but not on the same surface. The system may be pictured as three sets of threads imbedded in a solid.

. . . .

The physicist must assign times as well as places to his phenomena. This fact is of sufficient importance in the theory of relativity to require a special nomenclature. The position of an occurrence is called a *point*. Its position and time taken together are called a *point-event*, or shortly, an *event*.

To determine events every reference frame must be supposed to be filled with time indicating apparatus, called *clocks* for brevity, whatever their nature. Every point must theoretically have its clock. In practice we need suppose clocks to be only where they are wanted, just as lines in a reference frame were drawn

only where they were wanted. But wherever the clocks are, they must be set together and must go at the same rate, otherwise events could not be related definitely to one another.

The synchronization of two clocks is not so simple a matter as it might appear at first sight. If the two clocks are at the same place it can be perceived directly whether they are together or not. But if they are in different places some signalling method has to be arranged, and the meaning of synchronism will depend upon this system. If the clocks are of the usual construction, an obvious method is for the observer to place himself midway between them and arrange mirrors so that he can see their faces at the same time. That is to say, a system of light signalling is used. Accordingly, whatever the construction of the clocks, their indications are supposed to be signalled by light to a point midway between them, and if these signals reach an observer at that point simultaneously the clocks are said to be in synchronism. A succession of such observations determines whether they are going at the same rate or not. This definition of synchronism in terms involving light is an important point in the theory of relativity.

We shall call a reference frame with its clocks a *reference system*, and we shall use the words *observer's system*—omitting the word reference—to mean the observer himself, his reference system, his laboratory, and, in fact, all the objects which share his state of rest or motion.

. . . .

A difficulty is sometimes felt, not less by those who are accustomed to theoretical work in which co-ordinate systems are used than by others, in realizing that reference frames are anything more than paper diagrams. Paper diagrams are only representations of reference frames. In the sense in which the term is used in this book, a reference frame is to be taken as the systematized form of the ideas of distance and direction which every one is constantly using more or less vaguely and unconsciously. Thus the reference frame of a geographer consists of lines of latitude and longitude, the latitude of a place being measured by the angular distance of the place from the equator, and the longitude by the angle which a meridian circle through the north and south poles and the place makes with a similar circle drawn through some given place, such as Greenwich. The addition of sea-level gives him a complete three-dimensional reference frame. The reference frame of an astronomer is constructed on the same principle. There are three such in use. If, like the geographer, he takes the equator (or, rather, the trace of the plane of the equator produced to meet an imaginary celestial sphere) as his fundamental circle, he fixes any point by "declination" and "right ascension". Declination corresponds to terrestrial latitude, and right ascension to longitude, excepting that the fixed point corresponding to Greenwich is a point on the equator called the "first point of Aries," which is the place where the sun crosses the equator from south to north at the Vernal Equinox. The meridians of right ascension are drawn through the north and south poles. Right ascension is always

reckoned towards the east, and usually in hours, minutes, and seconds of time. Astronomers also use celestial latitude and longitude, but these do not correspond to terrestrial latitude and longitude. According to this system the fundamental circle is the ecliptic —the path of the sun in the heavens—and the meridians of longitude are drawn through the poles of this circle —that is, points 90° away from it. Within recent years a system of co-ordinates called " galactic latitude and longitude " has come into use. In these the Milky Way is used as a fundamental circle. A physicist uses all sorts of frames, the walls of his laboratory, or the arrangement of his instruments. The simplest measurement requires some sort of reference frame, or system if we include clocks. If it is only a matter of reading a mercury barometer or a thermometer a reference frame consisting of one line merely—the tube of the instrument—is used. Regarded in this way, a reference system is a real thing, and the origin may be taken as corresponding to what we call our point of view.

The subject matter of relativity is the correlation of aspects of things obtained from different points of view, and its object is to investigate the conditions under which it is possible to describe things in ways which will apply to different reference systems—that is to say, to inquire into the possibility of obtaining statements of fact which will hold good when one reference system is exchanged for another.

General rules exist for changing over, or transforming, as it is called, from one reference system to another, but the examples of tranformation which will concern

us are very simple, and they can be dealt with as they arise.

Summary.—Reference frames are means for locating positions in space. The most usual forms are Cartesian and Polar co-ordinates. Cartesian co-ordinates consist of intersecting sets of parallel straight lines or planes. Polar co-ordinates are practically the same thing as range, angle of sight, and bearing. Gaussian co-ordinates include the others as particular cases, and consist of a network of curved lines arranged continuously: they do not as a rule lend themselves to measurement. A reference system is a reference frame *plus* clocks. The clocks are synchronized by light signals. Reference systems are to be looked upon as really existing, and are not to be confounded with diagrams drawn on paper. A point of view is the origin of a reference system of some sort or other.

CHAPTER IV

VELOCITY, ACCELERATION, MASS, AND MOMENTUM

IN the subsequent work reference will be made to velocity, acceleration, mass, and momentum. We proceed to give short explanations of these terms.

1. Velocity

If a body moves through a distance l in time t, its average velocity throughout the time t is l/t. If the motion is uniform so that equal distances are described in equal times, this *average* velocity is the same as its *actual* or *instantaneous* velocity at any instant during the time t. If the motion is not uniform this is no longer the case, but the smaller the interval t the more nearly uniform is the motion and the more nearly does the average velocity throughout the interval approach the actual velocity at any instant during the interval. By taking time intervals short enough we can approximate as closely as we please to instantaneous velocities.

The term velocity implies more than mere speed; it involves the direction of displacement. Velocity is speed in a given direction, which direction must be specified. In this book we shall use the term *speed* instead of velocity when it is not required to take direction into account.

2. Acceleration

When the velocity of a body is changing either in magnitude or direction, the body is said to be accelerated, and any such change is always the result of the application of force. Without force there can be no acceleration. Every body possesses the property called inertia—that is to say, it persists in a state of rest, or of uniform motion in a straight line, unless acted on by a force.

The word acceleration is used in a technical sense in mechanics to cover not only increase, but decrease of speed with or without change of direction, and also change of direction without change of speed. If the force causing the acceleration is in the same straight line as the velocity of a moving body, the speed only varies but not the direction ; if it acts perpendicularly to the direction of the velocity, this direction is changed but no change of speed results. Intermediate directions of application of a force produce both effects combined.

If a body is rotating about an axis, every particle in it, excepting those on the axis, is subject to acceleration in the technical sense of the word, even though the rotation is uniform, since the direction of motion of each particle is continually changing though the speed may be constant.

Frequent mention will be made in what follows of accelerated systems of reference. This term will therefore be understood to include systems which are moving with varying speed without rotation, and also rotating systems in which the speed of the parts is not necessarily changing.

3. Mass

Mass is one of the fundamental physical quantities, like space and time, of which no satisfactory metaphysical definition can be given. It is sometimes said to be quantity of matter. If a piece of material possesses a certain mass, then a piece of the same material of double the volume will have double the mass under the same physical conditions. The masses of different bodies are proportional to their weights, if they are weighed in the same locality. Mass, however, is not the same thing as weight, for if a body be weighed at sea level and on a mountain top, or at the equator and one of the poles, its weight will differ in each case on account of its altered distance from the centre of the earth, though the quantity of matter remains the same.

4. Momentum

This is the name given to the product of the mass of a body into its velocity. Thus, if m be the mass of a body and v its velocity, the momentum is mv. As mass is sometimes said to be quantity of matter, so momentum is said to be quantity of motion, though the word *motion* is often used in the simple sense of displacement without any idea of mass attaching to it.

CHAPTER V

PHYSICAL LAWS

THE general physical laws of nature are statements in compact form of uniformities which experience has shown to exist amongst physical phenomena. They have no compelling or binding power like the laws of the land or Divine dispensations. They are simply generalized statements of what has been found to happen in given circumstances, and may therefore be expected to happen in the future in like circumstances.

Prior to the development of experimental science, which is of comparatively modern origin, physical laws were derived from metaphysical considerations. Thus the heavenly bodies were assumed by the ancients to move in circles, on the ground that nature was perfect and the circle was a perfect figure; no motion other than circular, therefore, was compatible with the perfection of nature. The absence of any adequate experimental means of checking physical laws threw the whole burden of their proof on to the soundness of their premises. Hence the supreme importance of metaphysical inquiry which alone, through pure reason, could be looked to for the groundwork.

The rise of the experimental method has altered all

this by making it possible to apply the method of hypothesis to the establishment of these laws. Some circumstance, or set of circumstances, makes it probable that a certain generalization holds good. It is, therefore, assumed provisionally as an hypothesis, and deductions are drawn from it. If these deductions, when tested by experiment, agree with observation, then the assumption reaches a higher degree of probability, which may rise to practical certainty if it gives an explanation of phenomena previously unexplained, and still more, if it has to its credit the prediction of new phenomena. For example, Newton assumed as an hypothesis his law of gravitation. He tried it on the moon, but owing to an inaccurate estimate of the moon's distance his calculations did not agree with observation. Some time afterwards, with the aid of more exact figures, he got concordant results. This went a long way towards establishing the theory. Newton himself, and many mathematicians after him, notably Laplace, applied the theory to the other bodies of the solar system, and it was found to explain practically all their movements, and even to predict the existence of an important new planet. The theory was then regarded as proved.

Its agreement with observation is astonishingly close. The only discrepancy of any importance is a small irregularity in the motion of the planet Mercury upon which it is not necessary to dwell at this stage, as it will be treated in greater detail later on. For the present it is enough to observe that Einstein's theory gives an adequate explanation, and it now seems clear that Newton's law of gravitation is only a first approximation to the truth, though an exceedingly close one.

The method of hypothesis has sometimes been stigmatized as mere guessing. This is unfair and foolish. An hypothesis is, of course, in the first instance very often a guess, but it is the sort of guess of which only talent and knowledge are capable, amounting at times to a flash of genius little short of inspiration. Those who use this language ignore the meticulous pains which are taken to verify by experiment the deductions from the hypotheses before they are accepted as laws. Einstein's theory is probably the finest instance on record of an inspired guess.

Natural laws do not "explain" anything in the widest sense of the word. They tell us what happens, but not how or why it happens. If when referring to physical laws such words as "cause," "because," "therefore," and the like, are used, no philosophical or metaphysical theory of the efficiency of causation is implied. A cause in physics is merely an antecedent set of circumstances found by experience invariably to precede another set which is called the effect. If, for example, we say that the velocity of a body is increasing *because* a force is acting, no reason is implied why forces should so act. All that is meant is that in past instances it has been found that acceleration is always preceded by the application of force, and to suppose that anything different is occurring in the present case is inconsistent with experience. Explanation, in the physical sense, is merely grouping together separate happenings into one general statement.*

* A very clear and full discussion of the scientific meaning of the word "explanation" is given in Herbert Spencer's "First Principles," Part I, Chapter IV.

The most important general feature of physical laws is that they are capable of mathematical expression; in fact, they require it. This distinguishes physical laws from such general statements as the law of supply and demand and other economic laws, Grimm's law, and various others belonging to the less exact sciences. Physical laws are used for exact deductions and numerical computation, and mathematical expression is essential. In this book, whenever reference is made to a general physical law, its mathematical expression will be understood.

As an example of the mathematical expression of a physical law, consider Newton's second law of motion, which is one of the postulates of mechanics. Newton himself calls it an "axiom". The law states that the change of motion of a body is proportional to the force impressed upon it, and takes place in the direction of the force. By "change of motion" is meant change of momentum, which was defined in the last chapter, and the "force impressed" means the product of the force into the time during which it acts. If, therefore, m be the mass of the body, v the velocity in the direction of the force F at the time when it commences to act, and v' the velocity at the end of a time t, then the change in momentum is $mv' - mv$, or $m(v' - v)$. This, the law says, is proportional to Ft, or equal to this product if units be properly chosen, so that

$$m(v' - v) = Ft$$

or,

$$F = m\frac{v' - v}{t}.$$

Now $(v' - v)/t$ is the change in velocity per unit

time, which, in continuation of what was said on the subject in the last chapter, may be defined to be acceleration. Call this acceleration f, and we have

$$F = mf$$

as the mathematical expression of Newton's second law. As a particular case, we may take that of a heavy body whose weight is W, and mass m. W is thus the force with which the earth attracts the body. If g is the acceleration, that is to say, the velocity, produced in one second by the earth's attraction, which is 32 feet per second, we have $W = mg$.

General physical laws are general in two ways. They must apply not only to large numbers of particular physical facts, but also to the circumstances of large numbers of particular observers. A law which is peculiar to the circumstances of one or a few observers only cannot be said to be general. Statements of physical laws must, therefore, as far as possible be independent of the points of view of particular observers; in other words, the forms of their mathematical expressions should be independent of any particular system of reference. This condition will be examined further in the next chapter.

Summary.—General physical laws are compact statements of uniformities. They are established by the method of hypothesis checked by experiment. They do not imply any metaphysical theory of causation. To be of any value they must be expressed mathematically. They should permit of statement in identical form for different observers.

CHAPTER VI

THE MECHANICAL PRINCIPLE OF RELATIVITY

PHYSICAL laws relate to measurable phenomena such as velocities, accelerations, forces, and the like located in particular places at particular times, and, therefore, in accordance with what we have seen in Chapter III, they require to be stated in relation to some reference system. In the last chapter we saw that for complete generality, they should be stated in such forms as are common to all observers. In the present chapter we shall inquire into the method of complying with this condition, confining the discussion to the laws of the Newtonian mechanics—the classical mechanics, as the subject has been called.

An observer's system must either be accelerated or unaccelerated, the latter term including a state of rest. We may at once rule out accelerated systems as unsuitable for the statement of general mechanical laws, for unless all the systems were subject to the same acceleration—a condition which is obviously impracticable—the different accelerations and the corresponding forces which we have seen always accompany them, would have to be taken into account, each system having accelerations and forces peculiar to it. Now, forces and accelerations enter into the statements of the laws

of Newtonian mechanics, as, for example, the second law of motion, which was considered in the last chapter. Consequently, if accelerated systems are used, the statements are complicated by accelerations and forces peculiar to each system. The possibility of framing any statement common to all is therefore precluded.

We have therefore to fall back on unaccelerated systems, and the question resolves itself into a choice of co-ordinates. To examine generally the various reference frames which present themselves would lead us too deeply into mathematics, and we shall therefore content ourselves for the present with saying that mechanical laws are usually stated with reference to *unaccelerated rectangular Cartesian co-ordinates*, or *Galilean co-ordinates* as they are called after Galileo, the founder of the modern science of mechanics. It has been found that when so stated general mechanical laws preserve their mathematical form whatever may be the relative motion between observers. In other words, when stated with reference to the system of one observer, they may be stated in exactly similar mathematical form with reference to that of any other moving relatively to him provided only that this motion is unaccelerated—that is to say, as the reader will remember, uniform in magnitude and direction and without rotation. We proceed to illustrate this, taking as an example Newton's second law of motion. For simplicity the investigation will be confined to one plane only, in which all the movements will be supposed to take place.

Suppose an observer stationed at *O* (Fig. 12), and

using the Cartesian reference frame Ox, Oy, to observe a particle P of mass m. By "particle" we understand a body which has mass, but the magnitude of which is so small that it need not be taken into account. Let the particle P be supposed to be already in motion parallel to Ox at the commencement of the observation, and let its velocity increase by an amount V during the time of observation t, under the influence of a force F, also parallel to Ox.

Let a second observer O', using the Cartesian reference system $O'x'$, $O'y'$ such that $O'x'$ slides along Ox with uniform velocity u, observe the same particle

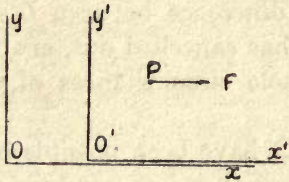

FIG. 12.

during the same time t, and let him apply the second law of motion to the particle. At the commencement of the time t we shall suppose that he observes its velocity to be v. Its momentum is therefore mv. At the end of the time its velocity, according to O', is $v + V$, since V, being a gain in velocity, is assumed to be the same for both observers. The momentum is, therefore, $m(v + V)$. Thus the change of momentum, according to the measurements of O', is $m(v + V) - mv$, or simply mV. This change the second law of motion declares to be equal to Ft, and thus

$$F = mV/t.$$

Now let O go through a corresponding process. O' is moving relatively to O with a velocity u, and since at the commencement of the time t the particle is moving relatively to O' with a velocity v, its velocity, according to O's measurements, is $u + v$, and its momentum is $m(u + v)$. Similarly, at the end of the time t, its momentum is $m(u + v + V)$, and its change of momentum is $m(u + v + V) - m(u + v)$ or mV. As in the case of O', O puts this equal to Ft, and thus obtains

$$F = mV/t,$$

which is exactly the same as the statement at which O' arrived. It will thus be seen that the velocity u, which constitutes the difference between O's point of view and that of O', has cancelled out, enabling them both to state the whole circumstances of the problem in identical form.

This would not have been possible had the relative velocity u between O and O' not been uniform. Suppose that in the time t it had altered from u to u', and also that according to O' the particle had gained a velocity V. Then the change of momentum, according to O', would be mV. But according to O it would be

$$m(v + V + u') - m(v + u)$$

or,

$$m\{V + (u' - u)\}$$

and O's statement of the second law would have been

$$F = \frac{mV + m(u' - u)}{t}$$

which is not identical with that of O'.

It is thus plain that the possibility of the existence

of identical statements of the law hangs upon the fact that the relative velocity is uniform. This is an essential condition.

The principle illustrated by this particular example is generally true, and we are thus enabled to act upon the following postulate :—

All Galilean reference systems are equally suitable for the statement of general mechanical laws.

This is what may be called the Mechanical Principle of Relativity. It will be observed that it is in reality not so much a principle of *relativity* as of *correlativity*, inasmuch as it points to the synthetic process of unifying different points of view rather than to the analytic operation of distinguishing between them.

We have now to inquire into certain assumptions which were tacitly made in the discussion on the second law of motion which has just been given.

It was assumed in the first place that the motion of O' relatively to O made no difference in the value of the time t in the reckoning of either observer. O evidently assumed that the clocks on the system of O' registered exactly the same time as those on his own, and conversely. Each observer, in fact, assumed that he might, had he wished, have made use of the other observer's clocks for measuring time, or, in other words, that a second appeared to be exactly the same to both.

In the second place, both of them assigned the same values to the velocities. For example, the velocity V which was gained by the particle was supposed by both to have the same value. It is true that, owing to the relative motion, the *total* velocity of P seemed different

to the two observers, but V was a *gain* in velocity, and was common to both. Now, it has been seen that a velocity is a comparison between length and time made in a special way, and since both observers believed their times to be the same, they must also have believed their lengths to be the same.

These assumptions, indeed, seem to be obvious common sense. For what difference, it may be asked, can the mere fact of movement make to the length of a rod? Why should a yard measure carried by either observer appear any different to the other merely because it is moving relatively to him, or why should a clock appear to alter its rate merely because it is moving? A railway passenger might as well expect his umbrella to shorten or lengthen, or his watch to gain or lose time when the train starts.

Nevertheless, obvious as they may appear, it must be kept in mind that they are but assumptions, and like all other assumptions, they must be judged by their consequences. The question is not, whether they agree with common sense, but whether deductions which involve them agree with experiment. If they do not they must be reconsidered.

As far as mechanics are concerned they have stood this test. The whole of physical astronomy, the predictions of which have been verified so abundantly,* rests upon them, and so do the calculations of engineers and others concerning bodies in motion. They must, therefore, be very largely if not absolutely true. For the moment we shall assume their truth and proceed to derive from them geometrical formulæ by which two

* Excepting as already noticed in Chapter V.

observers in relative motion can change over to each other's systems from their own—formulæ of transformation, as mathematicians call them.

Let P (Fig. 13) be a point referred to the same sets of axes which have already been considered in this chapter. Draw the ordinate PM, which is common to both systems, and the abscissæ PN, PN', which coincide since $O'x'$ is supposed to slide along Ox. Let O', as before, move with a velocity u relatively to O along Ox, and suppose that O and O' coincide at zero time. Let the figure represent the state of affairs at

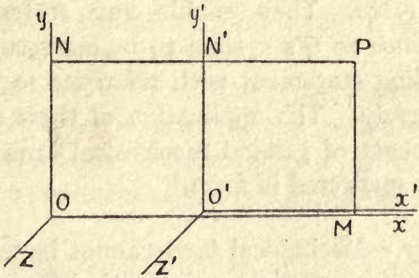

FIG. 13.

the end of time t according to the reckoning of O, and t' according to O'. Let the co-ordinates of P, PN and PM in O's system, be (x, y), and let the co-ordinates PN' and PM in that of O' be (x', y'). Since both systems coincide at zero time OO', or NN', is equal to ut. Thus PN', according to O, is $PN - OO'$, or $x - ut$. According to O', PN' is x', and since both observers, in agreement with one of the suppositions above discussed, ascribe the same value to PN', we must have $x' = x - ut$. PM is common to both observers, and we assume in a similar way that $y = y'$.

Had there been a third co-ordinate z, or z', we should similarly assume $z = z'$.

Again, since by the other supposition both observers ascribe the same value to the time, we shall have

$$t = t'.$$

We thus have

$$\left.\begin{aligned} &x = x' + ut \\ &(\text{or, } x' = x - ut) \\ &y = y' \\ &z = z' \\ &t = t' \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

as a set of formulæ which enable O to express his observed lengths and times in terms of those of O', and *vice versa*. They enable any statement made with reference to O's system to be converted into the corresponding statement with reference to that of O', and conversely. The application of these formulæ to the statements of general mechanical laws leaves the statements unaltered in form.*

Summary.—Mechanical laws cannot be expressed in similar form for systems moving with different accelerations. All Galilean systems are equivalent for the statement of mechanical laws. When we change over or transform a mechanical law from one Galilean system to another, it is assumed that each observer ascribes the same values to the other's lengths and times as to his own. Transformation formulæ for correlating the observations of length and time of two observers under these two suppositions are determined.

* The reader will observe that it is the *form* of the mathematical expressions which is preserved. Each observer states the facts in his own terms, e.g. x, y, z, t, or x', y', z', t', as the case may be. But all the statements agree as to the relationship in which these terms stand to one another.

CHAPTER VII

THE LORENTZ TRANSFORMATION

THE correlation between different points of viewing mechanical phenomena was seen in the last chapter to depend, according to the classical mechanics, on two suppositions, which we may state as follows in somewhat more general terms :—

(1) The time interval between two events is independent of the condition of motion of the reference system.

(2) The space interval between two points is independent of the condition of the motion of the reference system.

From these were derived the set of formulæ (1) of the last chapter, by which the suppositions could be applied to any given case of change of point of view. It was also seen that when so applied mechanical laws retained their form, so that these laws showed no preference for one system more than for another. These suppositions attribute an absolute character to space and time measurements which renders them independent of the motion of any particular observer.

But when these suppositions are applied to the general laws of electro-magnetic phenomena it is found that a preference is shown. If the laws of the agencies which act in an electro-magnetic field are stated with reference

to a system fixed with reference to the æther, which is the name given to the seat of those agencies or the medium in which they occur, and the transformation is applied to render the statement in terms of some other uniformly moving system, the velocity of this latter system does not cancel out as with mechanical laws, and the form of the statement is changed. As long as these suppositions are adhered to, a principle of relativity cannot be applied to electro-magnetics.

In order to meet this situation, and in accordance with certain electro-magnetic facts, Lorentz proposed the following scheme of transformation in place of the scheme of the last chapter :—

$$\left.\begin{aligned} x' &= \beta(x - ut) \\ y' &= y \\ z' &= z \\ t' &= \beta\left(t - \frac{ux}{c^2}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

Where u, as before, is the relative velocity between two systems, and

$$\beta = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}},$$

c being the velocity of light *in vacuo*. It will be observed that $\sqrt{1 - u^2/c^2}$ is less than unity, so that β, or $1/\sqrt{1 - u^2/c^2}$, is greater than unity. When this transformation is applied to the mathematical statement of the laws of the electro-magnetic field, it is found that the velocity u cancels out as in the case of mechanical laws, and the form of the statement is preserved.

Comparing the first of equations (1) and (2) we see that the Lorentz transformation involves the abandonment of the supposition that observers' estimates of lengths in each other's systems are unaffected by movement. For, referring to Fig. 13, $x - ut$ is O's estimate of the length PN', while x' is that of O'. The equation $x' = \beta(x - ut)$ means, therefore, that x' is greater than $x - ut$, since β is greater than unity. O's measurements of lengths in the system of O' are less than those of O' when these lengths are in the direction of motion. Electro-magnetic theory indicates the reality of this divergence. The equations $y' = y$ and $z' = z$ mean that lengths at right angles to the direction of motion are unaffected.

It might also be shown by comparison of the fourth equations of the two sets that the observers' estimates of times are likewise affected by the relative motion, so that the second supposition must also be abandoned, but as this point will be taken up later on, we shall not pause to consider it further at the present stage. We note, however, that the Lorentz transformation abandons the absolute character of space and time measurements and makes them dependent on the motion of each observer.

Though the Lorentz transformation was supported by electro-magnetic theory, it could hardly be regarded as satisfactory. The object with which it was proposed was to preserve the form of certain physical equations in changing over from one reference system to another, and it could scarcely be looked on as more than a mathematical device invented *ad hoc*. Einstein, however, showed that it followed at once without reference

to electro-magnetic theory from a remarkable property of the measure of the velocity of light which we proceed to discuss in the next chapter.

Summary.—The suppositions of the absolute character of space and time measurements do not enable electro-magnetic laws to preserve their form on transformation. Lorentz proposed a scheme which effects this, but which involves the abandonment of the absolute character of the measurements.

CHAPTER VIII

THE VELOCITY OF LIGHT

IT is proposed to examine in this chapter the two following postulates, and to draw from them a conclusion regarding the velocity of light relative to an observer. The postulates are :—

(1) It is impossible for anyone to determine his own absolute unaccelerated velocity.*

(2) The velocity of light *in vacuo* is independent of that of its source.

We shall consider these in order.

Every railway passenger must have experienced the difficulty of deciding whether his own train is moving, or an adjacent one, when the movement is slow. If he is moving and the movement is smooth and free from bumps or jolts, it is quite impossible to settle the matter by merely looking at the adjacent train. The passenger has to look out of the opposite window at some objects which he knows to be fixed relatively to the earth, such as buildings, or objects on the platform. In applying this test he cannot depend on any object on the train, that is, on his system ; he has to depend

* This postulate, or a statement to the same effect, is given in many books as a definition of the principle of relativity (restricted). The writer however has followed Einstein. See Preface.

on something outside his system, to tell him whether he is at rest or in motion. But if the carriage should jolt, even once, he recognizes his movement immediately. Now, a jolt or bump is a change in his motion, that is to say, an acceleration; consequently it is only when there is no acceleration, or when the acceleration is too slight for notice, that he has to look for external evidence. Failing noticeable acceleration there is nothing inside his compartment to tell him, since everything shares his state of rest or movement. But even if he has decided that he is actually moving, all he knows is that he is moving relatively to the earth. For all he knows the movement of the spot where the train is situated, due to the rotation and translation of the earth and the bodily movement of the solar system, may be such as exactly to cancel the motion of the train relatively to the earth, so that in point of fact he is at rest.

The same failure to discover absolute motion attends all mechanical experiments. As we have seen, mechanical laws have no preference for one uniformly moving system over another, and no distinction can be drawn between them. The fact is that there is no body of reference which is known to be fixed, and to which reference can therefore be made to determine the state of rest or movement of any other body. Without such a fixed body absolute motion is an unmeaning expression; only relative motion is determinate.

It was at one time thought that such a reference body could be found in the æther of space, using the word "æther" for the vehicle of transmission of light waves and other electro-magnetic radiations, such as heat or

wireless rays, without assigning any other properties to it. Certain experiments seemed to indicate that the æther was fixed, or, at all events, possessed only such motion as was shared by the whole visible universe, and should therefore be ignored. It was, therefore, hoped that electro-magnetic experiments—including experiments with light, which is known to be of electro-magnetic origin—would enable the earth's motion of translation to be determined with reference to it. But all these experiments failed.* No movement of any kind could be detected. It is, however, a long step from what experiment shows, namely, that no motion *has been* detected, to the statement of the postulate that no motion *can be* detected. It would, of course, be wholly unjustifiable if there were any reason to think that the experiments were of so clumsy a nature as to fail to detect a movement which really existed, or if there were any suspicion that all possible means had not been exhausted. So far from the experiments being clumsy, they were of so refined a character as to render possible the detection one-tenth or less of the expected result. The earth travels in its orbit at the rate of about 30 kilometres a second, and as the experiments were repeated at all times and seasons, the locality where they were conducted must have had this velocity

* The experiments were :—

Michelson-Morley, "American Journal of Science," 3rd series, Vol. 34 (1887), pp. 333-345 ; also, "Phil. Mag.," Vol. 24, 5th series, Dec., 1887 ; Morley and Miller, "Phil. Mag.," Vol. 9, 6th series, May, 1905 ; Trouton and Noble, "Proceedings R.S.," Vol. 72 (1903), p. 132 ; also "Phil. Trans.," Vol. 202 (1903), p. 165 ; Trouton and Rankine, "Proceedings R.S.," Vol. 80 (1907 and 1908), p. 420.

at least at some time. The experiments, however, would have detected a velocity of 3 kilometres per second. Moreover, mechanical and electro-magnetic means having failed, there is absolutely no other known agency available for experiment. The conclusion, therefore, seems unavoidable that the motion is undetectable, and we are, therefore, justified in adopting the postulate. What interests us in particular for present purposes is the fact that motion *relative to the medium which transmits light* is undetectable.

We now consider the second postulate, which states that the velocity of light is independent of that of its source. The meaning of this postulate can be made clear by an example. If a gun, whose muzzle velocity is 2000 feet per second, is fired in any direction from an armoured train at rest, the velocity of the shell will in all cases be the same. But if the train is moving at the rate of 15 miles an hour, or 22 feet per second, the velocity of the shell, if the gun is fired directly forward without elevation, will be 2022 feet per second, or if fired directly backward, 1978 feet per second. The velocity of the train affects the velocity of the shell to the extent of its own velocity of 22 feet per second plus or minus. The case of a ship at sea is wholly different. The waves due to the motion of the ship recede from it always at the same velocity independently of the speed of the ship. The only difference which the speed of the ship makes is in the sizes and lengths of the waves. So also with sound waves ; the velocity of a source of sound affects the lengths of the waves, but not their velocity when once started. In fact, the independence of the velocity of a wave and

its source is characteristic of wave motion generally. Now light, according to the wave theory, consists of waves, not in water or air, but in the æther, and light waves have the same property as others; their speed is always the same. The postulate is, therefore, a direct consequence of the wave theory of light, but it has been proved independently of any theory of light by observations on the fixed stars.*

The conclusion from these postulates is obvious. If the velocity of light in its medium is an absolute constant, and the observer cannot perceive his motion through that medium, it necessarily follows that the velocity of light relative to him must always appear to him to be the same and equal to its constant velocity in its medium. Or, we may put the matter in the form of a *reductio ad absurdum* thus: If the second postulate is accepted, and if in addition an observer could perceive any difference between the absolute velocity of light and his velocity relative to light, that difference would enable him to measure his own velocity in the medium, which is contrary to the first postulate.

As many people find considerable difficulty in accepting the proposition which has just been proved, some indeed considering it so preposterous as to amount to a *reductio ad absurdum* of the whole subject of relativity, it is necessary to examine its terms. A form in which it is sometimes stated, namely, that the velocity of light is the same for all observers, is certainly open to serious misconstruction. The statement really means that the velocity of light relatively to each observer

* Einstein, "Relativity," p. 17. Also "The Principle of Relativity," Saha and Bose (Calcutta University), pp. 172 *et seq.*

always appears the same *to him*, and equal to its constant velocity in its medium. It does not mean that the velocity of light relative to each observer appears the same *to other observers*. Thus, consider a light beam AB having its source at A, and let there be two observers O, O', the former of whom may be supposed to be stationary relatively to A, while the latter moves in the direction of the beam with a velocity u relatively to the former observer. The velocity of light *in vacuo* being 300,000 kilometres per second, O makes out the velocity relative to himself to be 300,000 kilometres per second, and O' also makes out the velocity relative to himself to be the same, but O computes the velocity of

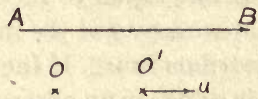

FIG. 14.

light relative to O' to be 300,000 kilometres per second minus u, and O' computes the velocity of light relative to O to be 300,000 kilometres per second plus u. The reason why O and O' make out the velocity to be the same, each relative to himself, is, as we shall presently see, because their relative motion affects their measurements of lengths and time. These measurements adjust themselves automatically in such a way as to give the same numerical value to the relative velocity in both their cases.

But, however we may regard it, the proposition is sufficiently strange and difficult to grasp at first. It obviously makes the velocity of light unique amongst all other velocities; the velocity of nothing else appears to every observer to be the same relative to himself.

It differs from all other wave motions because in every case, except that of light, the observer is able to recognize his movement through the medium in which the waves occur.

An example will illustrate the remarkable physical consequences of the proposition. Suppose two ships to pass at sea, going in opposite directions, and suppose that when they are abreast a splash is made in the sea between them. The waves spread out in all directions, and the ships go on, as it were, in pursuit, leaving behind them the place where the splash occurred. Neither ship is in any doubt about this, they are aware that they do not remain at the centre of the disturbance. But now suppose two observers to pass one another in space, and a light signal to be flashed between them. It might be supposed that the same thing would happen as in the case of the ships, but this is not so. As far as his observation can tell him, each observer thinks that the centre of the disturbance remains with him.

It is easily seen that the postulates imply that velocity of light must appear the same in all directions to any observer.

Summary.—Two postulates are stated and explained. The essential feature of the first is that no observer can detect his motion through the medium which serves as the vehicle for the transmission of light waves and other electro-magnetic radiations. The second is that the velocity of light through this medium is independent of that of its source. It is thence deduced that the velocity of light has the same measured value relative to every observer.

CHAPTER IX

THE RESTRICTED PRINCIPLE OF RELATIVITY

WE shall now show how the results of the last chapter affect the measurements of length of two observers moving with uniform relative velocity, and thence deduce the Lorentz transformation.* We shall then state and explain what is known as the Restricted Principle of Relativity.

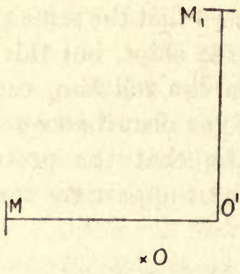

Fig. 15.

* The discussion which follows is not regarded by the writer as the simplest from a mathematical point of view, but the direct method involves more algebra than he is allowing himself. For this latter method, reference may be made to Einstein's book, "Relativity," Appendix I, or to "The Electron Theory of Matter" (O. W. Richardson), pp. 297-300. The method in the text was in part suggested by a paper by R. D. Carmichael in the "American Physical Review," Vol. XXXV, No. 3. Carmichael, however, applies the method to time, not length measurements.

Let us imagine two observers, O and O' (Fig. 15), in uniform relative motion with velocity u, but, as we have seen, neither can tell whether he is moving or not. O' provides himself with an apparatus consisting of two equal and mutually perpendicular arms $O'M$, $O'M_1$, at the ends of which are mirrors M and M_1, each perpendicular to its arm. The arm $O'M$ lies in the direction of relative motion. O' sends out light signals simultaneously along each arm from their intersection O'. These signals are reflected back, and since the arms are of equal length and the velocity of light the same in every direction, the signals return to their starting point in the same time and reach it simultaneously. Or, if the reader prefers, O' adjusts the lengths of the arms so that the signals return to him simultaneously, and he judges them to be equal. It does not matter how the adjustments are made; the essential point is that O' judges the arms to be of equal length, and the signals to take the same time over their journeys. All this takes place whether O' is moving or not. If his motion made any difference to him that difference would enable him to detect it, and this, we have agreed, is impossible.

Now O is looking on at these proceedings, and how does he regard them? He, like O', is unable to say whether he is moving or not. All he knows is that O', with all his instruments, is moving past him with a velocity u, as in Fig. 16. He sees O' sending signals along the arms, and he knows that O' receives them simultaneously. He may be supposed to know this either because he can see the instruments belonging to O', or because O' may have told him so. What he is not supposed to know is that O' thinks that the two

arms are of equal length, and he proceeds to compute their relative lengths from the fact that the signals perform their several journeys in the same time. It does not matter whether his reckoning of the time is the same as that of O', since it will be found to disappear from the equations. Provisionally, O calls the length of the arm in the direction of motion λ, and the other arm l, and he proceeds to relate l and λ by determining the times which the signals take over their journeys and equating them as follows :—

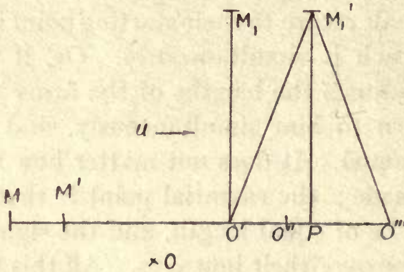

FIG. 16

I. THE SIGNAL IN THE DIRECTION OF MOTION

Let t_1 be the time the signal takes to go from the starting point O' up to the mirror M, which, by the time the signal reaches it, will have moved according to $O's$ reckoning from the place M, where it was when the signal left O' to some point M'. We shall thus have $O'M' = ct_1$, where c is the velocity of light. Now MM' is equal to the distance which both O' and the mirror have moved in the same time t_1 with their common velocity u, and therefore $MM' = ut_1$. Also, $O'M' = O'M - MM' = \lambda - MM'$, and therefore

$$ct_1 = \lambda - ut_1$$

or,
$$t_1 = \frac{\lambda}{c+u}$$

Meanwhile O' has, according to O, moved to O'', where $O'O'' = ut_1$.

We have now to consider the return of the signal to O''', which is the position of O' when the signal gets back to him. If t_2 be the time taken, we see that the light covers the distance $M'O'''$ in the time t_2, and thus $M'O''' = ct_2$. In the same way as before, we have $O''O''' = ut_2$, and $M'O'' = \lambda$.

Thus
$$ct_2 = \lambda + ut_2$$

or,
$$t_2 = \frac{\lambda}{c-u}.$$

Now T, the whole time for the out and home journeys, is $t_1 + t_2$, that is,

$$T = \frac{\lambda}{c-u} + \frac{\lambda}{c+u} = \frac{2c\lambda}{c^2 - u^2},$$

by the rule in algebra for the addition of fractions.

2. The Signal at Right Angles to the Direction of Motion

In this case the signal describes the two equal sides of the isosceles triangle $O'M_1'O'''$ in the time T—that is, each side is described in the time $T/2$; and since the velocity of light is the same in all directions, it is still c as before. Thus

$$O'M_1' = cT/2$$

where M_1' is the position of the mirror M_1 at the end of the time $T/2$. During this time O' has moved with

velocity u to P, where P is the foot of the perpendicular from M_1' on the direction of motion of O'. $O'P$, therefore, is $uT/2$. Now $M_1'P$ is what O calls l, and in the right angled triangle $O'M_1'P$

$$O'M_1'^2 = O'P^2 + M_1'P^2$$

by the theorem of Pythagoras,

or
$$\frac{c^2T^2}{4} = \frac{u^2T^2}{4} + l^2,$$

that is,
$$T = \frac{2l}{\sqrt{c^2 - u^2}}.$$

We therefore have two expressions for T which we can equate, and thus find

$$\frac{2\lambda c}{c^2 - u^2} = \frac{2l}{\sqrt{c^2 - u^2}},$$

or,
$$l = \frac{c\lambda}{\sqrt{c^2 - u^2}} = \frac{\lambda}{\sqrt{1 - \frac{u^2}{c^2}}}.$$

Thus, λ (the length of the arm in the direction of motion), which according to the reckoning of O' was equal to l (the length of the arm at right angles to the direction of motion), is, according to O, $l\sqrt{1 - \frac{u^2}{c^2}}$. That is, O computes it to be shorter, since $\sqrt{1 - \frac{u^2}{c^2}}$ is less than unity. This it will be remembered is exactly the contraction assumed in the equations of Lorentz.

Now suppose O' to be the spectator and O the experimenter. The circumstances are altered in no material respect. All that O' knows about the motion is that

O is passing him, and thus since both attribute the same velocity to light, a length measured in the direction of motion which O supposes to be l, is $l\sqrt{1 - u^2/c^2}$ according to O'. Each attributes a contraction to the other's lengths when these are measured in the direction of motion, and there is no means of deciding which determination is to be preferred.

Since u is small in ordinary cases compared with c, which is 300,000 kilometres per second, it will be seen that usually $\sqrt{1 - \frac{u^2}{c^2}}$, though less than unity, does not differ much from it, so that the contraction is small. Though the velocity of the earth in its orbit is 30 kilometres per second, the contraction in its diameter as seen from the sun would only amount to about $2\frac{1}{2}$ inches. The remarkable fact, however, is not the magnitude of the effect, whether small or great, but its occurrence. To produce a substantial contraction, an enormous relative velocity would be required. Let us find what relative velocity would produce a contraction of one-half. To do this we have only to put $\sqrt{1 - \frac{u^2}{c^2}}$ equal to one-half, and work out the value of u.

Thus $$\frac{1}{2} = \sqrt{1 - \frac{u^2}{c^2}}$$

or $$1 - \frac{u^2}{c^2} = \frac{1}{4}$$ This gives us

$$\frac{u^2}{c^2} = \frac{3}{4}, \text{ or } \frac{u}{c} = \frac{\sqrt{3}}{2} = \frac{7}{8} \text{ approximately.}$$

Taking c as 186,000 miles per second, this gives $u = 161{,}000$ miles per second approximately.

It should be observed that the contraction is attributed to a body in the direction of motion only. There is no evidence that it takes place in any other direction, and its absence is therefore presumed. This is, of course, an assumption, just as the absence of all alteration whatever was an assumption in the classical mechanics. The present assumption is, however, so far uncontradicted by any experiment, and there is no known phenomenon which suggests anything to the contrary, or which it would help us to explain. If in the future anything should come to light suggesting that movement affects lengths perpendicular to its direction the matter would, of course, have to be reconsidered, but until then such an assumption would be gratuitous.

We have now to deduce the Lorentz transformation. Referring to Fig. 17, let two observers, O and O', move relatively to one another with uniform velocity, u, so that the x-axes of both coincide. For simplicity we shall, in the first instance, consider two spatial dimensions only—that is to say, all the phenomena will take place, and the measurements will be supposed to be made, in the plane of the paper. The extension to three spatial dimensions is easy.

Let P be any point fixed in the system of O'. Draw PM perpendicular to Ox, or $O'x'$, and $PN'N$ perpendicular to $O'y'$, or Oy, to meet these two lines in N' and N. Let $PM = y$, or y', and $PN' = x'$, while $PN = x$. Let us suppose that O' coincides with O at zero time, and that the system of O' has come into the position shown in the figure after a lapse of time t, according to the reckoning of O, and t', according to

the reckoning of O'. Then $OO' = ut$, and PN', or $PN - OO'$, according to O's reckoning is $x - ut$. This is the distance which O' calls x'. But by the result which has just been obtained this distance appears to O to be $x'\sqrt{1 - \frac{u^2}{c^2}}$,

that is $$x'\sqrt{1 - \frac{u^2}{c^2}} = x - ut.$$

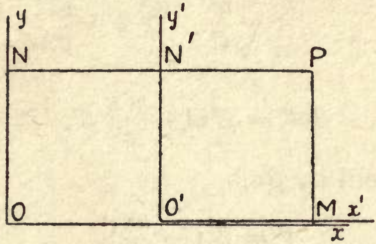

FIG. 17.

Thus $$x' = \frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}}$$

or $$x' = \beta(x - ut) \quad . \quad . \quad . \quad (3)$$

where β is written for $\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$ for brevity.

We may relate x and x' in terms of t' in the same way. PN is equal to $PN' + NN'$. PN' as before is x', according to the reckoning of O', and NN' is ut'. Thus the distance PN is $x' + ut'$ in the reckoning of O'. This is the distance which O calls x, and it therefore appears to O' to be $x\sqrt{1 - u^2/c^2}$. We thus get

$$x\sqrt{1 - u^2/c^2} = x' + ut',$$

or $$x = \beta(x' + ut'). \quad . \quad . \quad . \quad (4)$$

By means of (3) and (4) we can relate t and t' in terms of either x or x'. Substituting for x' in (4) the value $\beta(x - ut)$ given by (3) we get

$$x = \beta\{\beta(x - ut) + ut'\}$$
$$= \beta^2 x - \beta^2 ut + \beta ut'.$$

Thus $$\beta ut' = x(1 - \beta^2) + \beta^2 ut.$$

Now $$1 - \beta^2 = 1 - \frac{1}{1 - \frac{u^2}{c^2}} = -\frac{\frac{u^2}{c^2}}{1 - \frac{u^2}{c^2}} = -\frac{u^2\beta^2}{c^2}.$$

Thus $$\beta ut' = \beta^2 ut - \frac{u^2 x}{c^2}\beta^2,$$

or, dividing out by βu,

$$t' = \beta\left(t - \frac{ux}{c^2}\right). \quad . \quad . \quad . \quad (5)$$

In a similar way we might have obtained

$$t = \beta\left(t' + \frac{ux'}{c^2}\right) \quad . \quad . \quad 5(a)$$

by substituting for x in (3) the value $\beta(x' + ut')$ given by (4).

Since the relative movement does not affect lengths at right angles to it, we have

$$y = y'.$$

To introduce a third dimension, all we have to do is to move P, N' and N out of the plane of the paper towards the reader through a distance z or z' in the obvious way shown in Fig. 18. z, like y, is perpendicular to the direction of motion, and therefore

$$z = z'.$$

Collecting all these results we have

$$\left.\begin{aligned} x' &= \beta(x - ut) \\ y' &= y \\ z' &= z \\ t' &= \beta\left(t - \frac{ux}{c^2}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (6)$$

and

$$\left.\begin{aligned} x &= \beta(x' + ut') \\ y &= y' \\ z &= z' \\ t &= \beta\left(t' + \frac{ux'}{c^2}\right) \end{aligned}\right\} \quad . \quad . \quad (7)$$

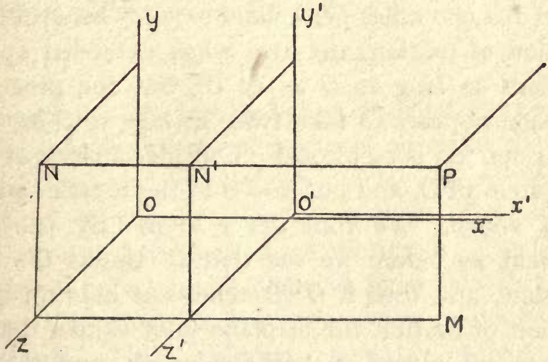

FIG. 18.

We are thus brought by perfectly general considerations back to the equations of the Lorentz transformation, which are shown in this manner to be independent of any electro-magnetic phenomena, and to be consequences of relative motion pure and simple.

We have seen that the observers ascribe a contraction to each other's lengths ; let us now see how they regard each other's times. Consider first a clock at O' on the

system of O', and let us see how O regards it. Since the clock is at the origin O', we have $x' = 0$. Thus from the fourth equation of (7) above we have

$$t = \beta t'.$$

Now t' is the time indicated on this clock according to O', and t is the time indicated on the same clock according to O. β is greater than unity, so that in the opinion of O a longer time has elapsed since zero time than in the opinion of O'. Thus all movements on the system of O' appear to O to be more lethargic than to O'. If, for example, an airplane pilot passing O at the rate of 161,000 miles per second extends his arm in the direction of motion, the arm when extended appears only half as long to O as to O', but the process of extension appears to take twice as long to O as to O'.

This opinion is reciprocal. Consider a clock at O on the system of O, and put $x = 0$ in the fourth equation of (6) above. We thus get $t' = \beta t$. By the same argument as before we see that O' thinks O's clock goes slow, and thus if O stretches out his arm in the direction of motion the airplane pilot thinks that the arm has contracted to half its length, and that the extension takes twice as long, compared with O's reckoning. O and O' each think that the lengths of the other have contracted and their times lengthened.

By the use of equations (6) or (7) we can transform or change over from the point of view of one observer to that of another, while in so doing, as was seen in Chapter VII, electro-magnetic laws preserve their form, the observers being supposed to use Galilean frames of reference. We are thus able to say that

All Galilean frames of reference are equally suitable for the statement of general physical laws.

This is called the Restricted Principle of Relativity. It is called "restricted" because its application is confined by the terms of the definition of a Galilean frame of reference to unaccelerated systems.*

Finally, the reader should note that the equations (6) and (7) are merely the embodiment in form suitable for transformation of the different estimates which observers make of each other's lengths and times, just as equations (1), Chapter VI, embody the suppositions that their estimates are the same.

Summary.—The fact that the velocity of light relative to every observer is the same causes observers to ascribe contractions to lengths on each other's systems measured in the direction of motion. Lengths perpendicular to this direction are unaltered. The Lorentz transformations can be deduced from this fact. Observers also think each other's times are longer than their own. These different estimates of length and time render possible the statement of a principle of relativity which includes all physical laws, and they are embodied in the formulæ which enable observers to change over from one point of view to another.

* It is also called the "special" principle for the same reason.

CHAPTER X

SOME SPECIAL FEATURES OF THE RESTRICTED THEORY OF RELATIVITY

THE purpose of this book being to explain the principles upon which differently circumstanced observers can state their facts in general form so that they can all tell the same story, the last chapter strictly speaking brings us to the end of what has to be said on the subject of the Restricted Principle of Relativity. The whole theory arising from the application of this principle to physical laws generally is beyond our purpose. Indeed, the development of the General or Gravitational Theory of Relativity, which will be dealt with in due course, has robbed the restricted theory of much of the interest which it originally possessed, excepting in the field of electro-magnetics. The General Theory has no logical dependence upon the restricted theory, as will be seen ; but the restricted theory is an almost indispensable introduction to it. A geometrical development to which it leads is, in fact, essential to the general theory. This development will form the subject of the next chapter, but meanwhile we give, by way of digression, some results of such interest or importance as to require reference.

1. Mechanics

The Newtonian laws of mechanics, having been originally stated with reference to Galilean systems under the suppositions of unalterable lengths and times as between differently circumstanced observers, naturally require modification if stated in accordance with the suppositions implied in the transformation of Lorentz. It is impossible to do more than refer to this matter here, but something further of a general character will be said on the subject in the next chapter. To give even a summary of the particulars would mean a treatise on mechanics, and in addition to this, since mechanics and electro-magnetics are now so closely connected, a greater knowledge of this latter subject is required than all the readers of this book can be assumed to possess.*

2. Simultaneousness of Events and Rates of Clocks.

Let two separate events take place at points $(x_1\, y_1\, z_1)$ $(x_2\ y_2\ z_2)$ and times t_1 and t_2 in O's reckoning, and $(x_1'\, y_1'\, z_1')$ $(x_2'\, y_2'\, z_2')$ and times t_1' and t_2' in the reckoning of O'. Then by the fourth equation of (6), Chapter IX, we have

* Those who wish to follow up this matter will find it developed in such books as the following: "The Theory of Relativity," by R. D. Carmichael (Chapman and Hall); "The Principle of Relativity," by E. Cunningham (Cambridge University Press); "The Theory of Relativity," by L. Siberstein (Macmillan). The first of these is the simplest.

$$t_1' = \beta\left(t_1 - \frac{ux_1}{c^2}\right)$$

$$t_2' = \beta\left(t_2 - \frac{ux_2}{c^2}\right)$$

Subtracting the first of these equations from the second we get

$$t_2' - t_1' = \beta(t_2 - t_1) - \beta\frac{u}{c^2}(x_2 - x_1).$$

We can draw two conclusions from this :—

(1) If the events are simultaneous according to O,

$$t_2 = t_1, \text{ or } t_2 - t_1 = 0.$$

Thus $$t_2' - t_1' = -\beta\frac{u}{c^2}(x_2 - x_1).$$

Now, if the events are also simultaneous according to O', t_2' must be equal to t_1'. But this cannot be the case unless $x_2 = x_1$. By using the fourth equation of (7), Chapter IX, we can show in the same way that x_1' and x_2' must be equal. The x-distances of the two events must, therefore, be the same to both observers—that is to say, the events must occur at some place situated in a plane perpendicular to Ox.

(2) Suppose the events to occur at the same place. Then $x_1 = x_2$ and

$$t_2' - t_1' = \beta(t_2 - t_1).$$

Now let the events be two successive beats of a clock which beats seconds. Then $t_2 - t_1$ is a second according to O's reckoning, and $t_2' - t_1'$ is a second according to that of O'. Now β is always greater than unity, and therefore a clock beating seconds on the system of O' appears to O to beat more slowly than one on his own—

that is, the clocks on the system of O' appear to O to go at a slower rate than those on his own. This discussion amounts to the same thing as that given at the end of Chapter IX.

If O' were to move with the velocity of light $\dfrac{1}{\sqrt{1-\dfrac{u^2}{c^2}}}$ would become infinite. Thus an observer at O would see the pendulum of the clock at O' at the commencement of a swing, but he would never see it reach the other end. The clock would appear to O to have stopped altogether. As it is not desirable to introduce considerations of the velocity of sound, we have pre-

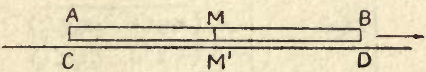

FIG. 19.

ferred the above form of statement to saying " O would hear the first beat of the clock at O', but he would never hear the second beat ". But it comes to the same thing.

We give an illustration, due to Einstein, to show that events which appear simultaneous to one observer are not necessarily so to a second observer moving with a relative velocity.

AB (Fig. 19) is a train moving in the direction of the arrow. When A and B are opposite points C and D on the permanent way, flashes of lightning occur at C and D, and will be judged to be simultaneous by an observer standing by the line at M', the point midway between C and D. Let M be the middle point of the

train, which is opposite M' when the flashes occur. Now, an observer at M is travelling towards D and away from C. Consequently he is meeting the light coming from D, and moving away from the light coming from C. The flashes will, therefore, not appear to be simultaneous to him. It is hardly necessary to state that since either observer can consider himself at rest, the above effects are reciprocal. For example, each observer thinks that the other observer's clocks go slower than his own.

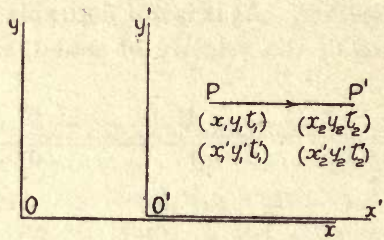

FIG. 20.

3. VELOCITY AND ACCELERATION

(1) *Velocity in the Direction of Motion,* i.e., *parallel to the x-axis*

Let u_1' be the velocity of a point P (Fig. 20) relatively to O' in the direction Ox (or Ox'). Let x_1' and x_2' be the x-distances of P at times t_1' and t_2' respectively. Then P moves relatively to O' through a distance PP', or $x_2' - x_1'$ in a time $t_2' - t_1'$, so that

$$u_1' = \frac{x_2' - x_1'}{t_2' - t_1'}.$$

We have to find the velocity of P as it appears to O. If we call this velocity u_1 we shall have

$$u_1 = \frac{x_2 - x_1}{t_2 - t_1},$$

where the x's and t's correspond in O's reckoning to the x''s and t''s given above in that of O'.

Substituting for x_1, x_2, t_1, t_2 from the first and fourth of equations (7), Chapter IX, we have

$$u_1 = \frac{\beta(x_2' + ut_2') - \beta(x_1' + ut_1')}{\beta\left(t_2' + \dfrac{ux_2'}{c^2}\right) - \beta\left(t_1' + \dfrac{ux_1'}{c^2}\right)}.$$

The β's all cancel, and we have

$$u_1 = \frac{x_2' - x_1' + (t_2' - t_1')u}{t_2' - t_1' + \dfrac{u}{c^2}(x_2' - x_1')}.$$

Dividing the numerator and denominator by $t_2' - t_1'$ we get

$$u_1 = \frac{\dfrac{x_2' - x_1'}{t_2' - t_1'} + u}{1 + \dfrac{u}{c^2}\,\dfrac{x_2' - x_1'}{t_2' - t_1'}} = \frac{u + u_1'}{1 + \dfrac{uu_1'}{c^2}}.$$

This result states the addition theorem for velocities according to the relativist view. In the classical mechanics, if a point is moving with a velocity u, and a second point is moving relatively to the first with a velocity u_1' in the same direction, the velocity of this second point is $u + u_1'$. Thus, if a train is moving at the rate of 40 miles per hour, or 66 feet per second, and a passenger walks along the corridor at the rate

of 3 feet per second in the same direction, his velocity relatively to the ground, according to the classical view, is 69 feet per second. According to the relativist view it is

$$\frac{69}{1 + \frac{66 \times 3}{c^2}} \text{ feet per second,}$$

where c is the velocity of light expressed in feet per second, which is somewhat less.

It is interesting to see what the result is when u_1' is the velocity of light—that is to say, if a point is moving with a velocity u relatively to an observer who considers himself fixed, and a light beam is sent out in the same direction from the relatively moving point, what is its velocity with reference to the "fixed" observer? Putting $u_1' = c$, we have

$$u_1 = \frac{u + c}{1 + \frac{u \cdot c}{c^2}} = c.$$

Thus the velocity relative to the fixed observer is also c. This is not a new result, as it was the basis of that of the last chapter, from which the present results have been derived, but it is interesting to note that the present result is consistent with the previous one.

Lest the reader should think that we are trying to bewilder him with paradoxes, it may be well to remind him that we are speaking all the time of physical measurements. It is on points such as the present that a person is apt to lose himself by unconsciously importing metaphysical ideas of extension and duration. What the statement which has just been made means,

is that if anyone actually measures the velocity of light it will always relatively to himself figure out to the same number. Similarly with the statement that no velocity can exceed that of light, which is also true in the physical sense. This may be proved in many ways, the present amongst them. We have just tried to add a velocity u to that of light, and we find that the result comes out to be the velocity of light over again. What we mean, is that if there is such a velocity there is no way of recognizing it, for no means exist for measuring it.

(2) *Velocity at Right Angles to the Direction of Motion*

Let v_1' be the velocity of P relatively to O' in the direction Oy (or Oy'), and let y_1' and y_2' be the y-distances at the times t_1' and t_2'. Let v_1, y_1, and y_2 correspond in O's reckoning to v_1', y_1', and y_2' respectively. Then

$$v_1 = \frac{y_2 - y_1}{t_2 - t_1} = \frac{y_2' - y_1'}{\beta\left(t_2' + \frac{ux_2'}{c^2}\right) - \beta\left(t_1' + \frac{ux_1'}{c^2}\right)}$$

$$= \frac{1}{\beta} \cdot \frac{y_2' - y_1'}{t_2' - t_1' + \frac{u}{c^2}(x_2' - x_1')}$$

$$= \frac{1}{\beta} \cdot \frac{\dfrac{y_2' - y_1'}{t_2' - t_1'}}{1 + \dfrac{u}{c^2} \cdot \dfrac{x_2' - x_1'}{t_2' - t_1'}} = \frac{1}{\beta} \cdot \frac{v_1'}{1 + \dfrac{uu_1'}{c^2}}.$$

Similarly, if w_1 and w_1' are the relative velocities in the direction Oz

$$w_1 = \frac{1}{\beta} \cdot \frac{w_1'}{1 + \dfrac{uu_1'}{c^2}}.$$

Here it is to be noticed that although lengths perpendicular to the direction of motion appear the same to both observers, velocities perpendicular to the direction of motion do not. We see, also, that the parallelogram of velocities does not hold good in the classical form. According to the classical view, if a point is moving with a velocity u relatively to an observer O, and another point is moving in a direction at right angles with a velocity v_1' relatively to the first point, the resultant velocity is represented by the

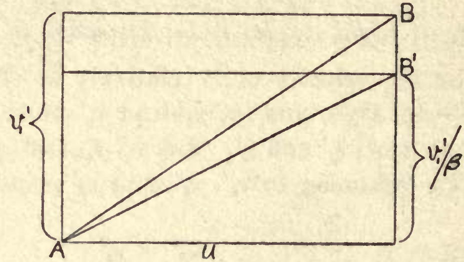

FIG. 21.

diagonal AB (Fig. 21) of a rectangle, the sides of which are proportional to u and v_1'—that is, the resultant is

$$\sqrt{u^2 + v_1'^2}.$$

According to the relativist view the resultant is AB', and is equal to

$$\sqrt{u^2 + \frac{v_1'^2}{\beta^2}}$$

relatively to O.

We might pursue the same line of argument with respect to accelerations, but this matter is not of

immediate interest. The results, as might be expected, are of the same form as for velocities.

4. Mass

According to the classical view, mass has always been held to be the same for the same body under all conditions of motion. We shall now inquire whether relative motion will affect its measure in like manner to the measures of length and time.*

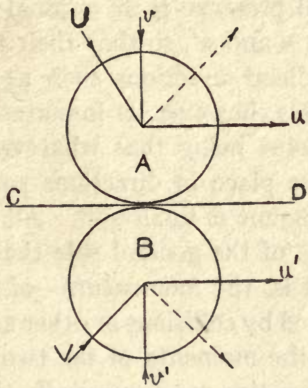

FIG. 22.

As a preliminary, let us consider the following case. Referring to Fig. 22, let two smooth, perfectly elastic spheres, A of mass m and B of mass m', moving in straight lines with velocities U and V respectively in any direction, collide. The size of the spheres is immaterial. By a process which is well known, and which the reader may take for granted, we may replace

* The substance of this section is taken from Carmichael's book "The Theory of Relativity" (p. 49), cited above. See also Lewis and Tolman, "Phil. Mag.," 18, pp. 510-523

the velocity U by two others, v and u, v perpendicular to the tangent CD to both spheres at the point of collision, and u parallel to it. The velocity V may be replaced by two velocities, v' and u', in a similar way. Now the laws of mechanics tell us that the collision does not affect the velocities u, u', but in the direction perpendicular to CD the velocities are modified, though in such a way that after collision the total amount of momentum in this direction is the same as before. The spheres still preserve their original momenta due to the velocities u and u', so that their total momenta are in some inclined directions such as are shown by dotted lines ; this, however, is immaterial for present purposes, the point being that whatever exchange of momentum takes place in directions perpendicular to CD, its total amount is unaltered. All this is merely a particular case of the general rule that the quantity of motion—that is, the momentum—of any system of bodies is unaltered by collisions or other actions between the bodies. If the momenta of the two bodies in this perpendicular direction are numerically the same before collision the exchange will not affect the magnitude of either ; it will simply reverse its direction.*

Referring to Fig. 23, suppose two observers to be provided with spheres of equal mass and size. We will imagine the observers to have met and compared the spheres, so that when the comparison is being made no question of relative velocity arises. We next suppose the observers to separate and move off some-

* See any book on dynamics, *e.g.*, Tait's "Dynamics," pp. 198 and 199.

where into space, and to acquire a velocity u relative to each other. One of them, O, may consider himself fixed, and regard O' as moving past him with the velocity u. Each projects his sphere at right angles to the direction of motion and with the same velocity v, each according to his own reckoning, in such a way that when O and O' are directly opposite one another the spheres collide exactly midway at A. This means of

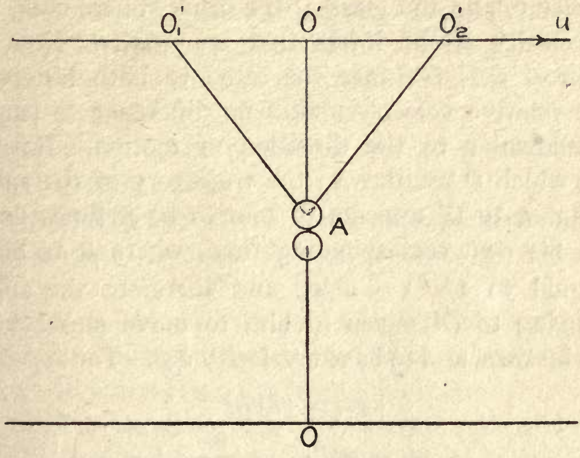

FIG. 23.

course that O' will have to project his sphere at some place O_1' slightly before he comes opposite to O. The sphere belonging to O' will have, in addition to v, the velocity u in the direction of motion. It will preserve this velocity u after collision, since the spheres are smooth, and it will return to O', whom it meets when he has reached a point O_2', such that $O'O_2' = O'O_1'$, so that the momentum due to this velocity is the same as before. The sphere simply retains it. The spheres

interchange the other velocities, but since these are the same, the exchange makes no difference, and each returns to the thrower with the same velocity v. Now what does O infer from the fact that his sphere returns to him with the same velocity as that with which he projected it? He infers that the other sphere must have had the same momentum as his own. If in his reckoning the mass of his own sphere is m, and its velocity v, and the mass of the other sphere is m_1 and its velocity v_1, he infers that $mv = m_1 v_1$. Now the distances OA, $O'A$ are the same to both observers, since relative velocity makes no difference to lengths perpendicular to the direction of motion. But the time which O ascribes to the trajectory of the sphere belonging to O' appears to him to be β times longer than his own corresponding time, where β as before is equal to $1/\sqrt{1 - u^2/c^2}$, and therefore the sphere belonging to O' seems to him to move more slowly than his own, and to have a velocity v/β. Thus $v_1 = v/\beta$

and $$mv = m_1 v/\beta$$

or $$m_1 = m\beta = \frac{m}{\sqrt{1 - u^2/c^2}}.$$

Thus the sphere belonging to O' appears to have increased in mass since the time when the observers made their comparison under the same conditions.

As before, this opinion is mutual. O' thinks that O's sphere has increased in mass compared with his own, though when they compared the two spheres together the masses were the same.*

* The foregoing discussion relates to what is called "transverse" mass; that is, mass measured transversely to the direction of relative

This result is of considerable importance, as the negatively electrified particles called electrons which are ejected from radio-active substances exhibit changes in mass. Since the velocities of the electrons in such cases may be of the order of that of light, these changes may become observable.

Summary.—(1) Mechanical laws require restatement, in view of the suppositions of variable lengths and times which underlie the Lorentz transformation.

(2) Events which appear simultaneous to one of two observers in relative motion are not generally simultaneous to the other.

(3) Under the above circumstances each observer thinks that the other observer's clocks go slower than his own.

(4) Velocities and accelerations in the same direction cannot be compounded by the simple process of adding them or subtracting one from the other.

(5) The parallelogram of velocities (and of accelerations) does not apply in the form stated in the classical mechanics.

(6) Mass appears to increase with velocity.

motion. Mass measured in the direction of this motion is called "longitudinal" mass. It is subject to yet another change, but it is of no interest or importance. As it can only show itself in the direction of motion, and an enormous velocity would be required to make it measurable, it cannot be made the subject of experiment.

CHAPTER XI

THE FOUR-DIMENSIONAL CONTINUUM

WE shall discuss in this chapter the geometrical implications of the Lorentz transformation. We shall first consider the case of two dimensions of space only.

Let P and Q (Fig. 24) be any two points whose

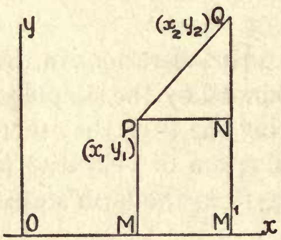

FIG. 24.

positions are determined by reference to some rectangular frame, Ox, Oy. Let the co-ordinates of P and Q be $(x_1 y_1)$ $(x_2 y_2)$ respectively. We proceed to find an expression for the length PQ in terms of these co-ordinates. Draw PM, QM' perpendicular to Ox, and PN perpendicular to QM. Then, since PQN is a right-angled triangle, we have by the theorem of Pythagoras

$$PQ^2 = PN^2 + QN^2.$$

But $$PN = OM' - OM$$
$$= x_2 - x_1$$

and $$QN = QM' - NM'$$
$$= QM' - PM$$
$$= y_2 - y_1$$

therefore $$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Now whatever axes we use to locate P and Q, provided we keep to the same plane, an expression of the form

$$(x_2 - x_1)^2 + (y_2 - y_1)^2$$

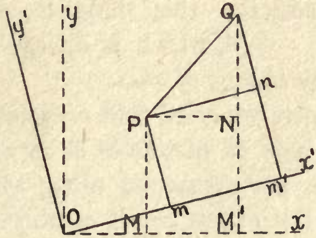

FIG. 25.

will always have the same value, since it always represents the distance between P and Q, and this distance does not depend upon the choice of axes. Take, for example, new axes Ox', Oy' (Fig. 25), inclined to the original ones, which are shown in dotted lines. If $(x_1'y_1')$ $(x_2'y_2')$ be the new co-ordinates, and we make the same construction as before, represented by reference letters m, m' and n, corresponding to M, M' and N, we shall have

$$PQ^2 = Pn^2 + Qn^2$$
$$= (x_2' - x_1')^2 + (y_2' - y_1')^2.$$

In the same way we may try any other set of rectangular of axes in the same plane (we need not even keep to the same origin), and we shall find that the expression having the form

$$(x_2 - x_1)^2 + (y_2 - y_1)^2$$

always preserves the same invariable magnitude whatever set of axes may be used.

An expression of this kind is called an *invariant.* This does not necessarily mean the same thing as a *constant,* for the distance PQ may be anything we like; but what we mean by invariant is that once having selected this distance, the expression $(x_2 - x_1)^2 + (y_2 - y_1)^2$, which is always equal to it, is *unaltered by any change of axes.*

In what follows we shall find occasion to apply this result to all kinds of distances such as PQ, whether these distances are measured along straight lines or curves. Now the above result is only strictly true if PQ is straight, but if we stipulate that PQ is always to be taken as very small, it will be *substantially straight whether it forms part of a curve or not.* A curve may, in fact, be regarded as the limiting case of a polygonal figure whose sides are infinitesimally short. It may be imagined as made up of a series of elementary straight parts placed end to end so that each *element,* as it is called, is inclined at an infinitesimal angle to the preceding one.

With this understanding the following notation is adopted. It is usual to represent the length of any arc of a curve measured from some fixed point A by the letter s. Thus in Fig. 26 we might call the arc

AP, s_1 and the arc AQ, s_2. PQ is therefore $s_2 - s_1$. When PQ, according to our stipulation, is small enough to be regarded as straight, we agree to express this fact by calling it ds. Thus ds means an element of arc; or, in fact, any elementary length. If, dropping the suffix, we call AP, s, then ds is the increment of s, or the elementary length which has to be added to AP to bring us to the adjacent point Q. The symbol simply means a small change in s. If (x, y) be the co-ordinates of P, the corresponding changes PN and QN in x and y are written dx and dy, conformably with

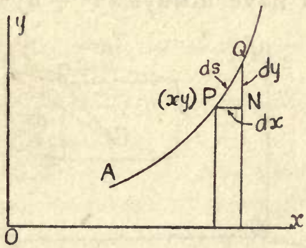

FIG. 26.

the notation ds for the length PQ. Thus if ds represents an elementary length in any direction, and dx and dy represent the corresponding elementary lengths measured parallel to the axes of reference, we have

$$ds^2 = dx^2 + dy^2$$

as the equivalent of

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

when all the distances are small, and PQ can therefore be considered straight.

It is to be noted that there is no special significance in the notation ds beyond the fact that ds means a very

short straight length. The symbol, for example, does *not* mean d multiplied by s. The letter d may be read as the initial letter of the word " difference ". ds, dx, and dy may conveniently be called *line elements*. ds is the general expression for a line element in any direction, dx and dy are line elements parallel to the axes. The relation

$$ds^2 = dx^2 + dy^2$$

means that $dx^2 + dy^2$ is the equivalent in any reference frame of the square of the line element in any direction. If we represent any change of axes by dashed letters x', y', we shall have always $dx^2 + dy^2 = dx'^2 + dy'^2$.

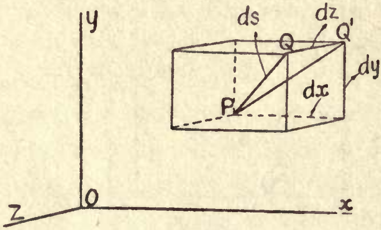

FIG. 27.

We note (1) that the expression $dx^2 + dy^2$ refers to a frame of reference in *two* dimensions of space only; (2) that it consists of *two* terms, dx^2 and dy^2; (3) that it is invariant.

Next consider the case of three dimensions of space.

Take any reference frame consisting of rectangular axes Ox, Oy, Oz, and let PQ be a line drawn in any direction. Let the co-ordinates of P be $(x_1 \, y_1 \, z_1)$ and those of $Q(x_2 \, y_2 \, z_2)$. Then, as the reader can satisfy himself,*

* If the reader has any difficulty about this statement he is advised to make a paper model. His difficulties will then disappear.

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

or, with the same stipulation as before,

$$ds^2 = dx^2 + dy^2 + dz^2.$$

We note that $dx^2 + dy^2$ now represents the square of PQ', the projection of PQ on the plane $x\,Oy$, and it therefore varies in general with the inclination of this plane with reference to PQ. It depends upon the planes of reference chosen, and is therefore not orientation of the invariant. The invariant in this case is the full expression for the square of the line element ds, or

$$dx^2 + dy^2 + dz^2,$$

which consists of *three* terms corresponding to the *three* spatial dimensions.

Comparing the cases of two and three dimensions of space we see that *the invariant expression for the square of the general line element contains as many terms as the number of dimensions of space under consideration.* This suggests that if any transformation of co-ordinates (or reference frame, or point of view, as the reader prefers) introduces additional quantities and corresponding additional terms into the expression, we know that we have introduced as many additional dimensions. We shall now apply this to the Lorentz transformation. Let us take the expression for the square of the distance PQ between two points

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2,$$

and test it for invariance by applying the appropriate formulæ of set (7), Chapter IX—that is, we put

$$x_1 = \beta(x_1' + ut_1),$$
$$y_1 = y_1',$$
$$z_1 = z_1',$$

and make similar substitutions for x_2, y_2, and z_2. We then get

$$\begin{aligned}&(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \\ &= \{\beta(x_2' + ut_2') - \beta(x_1' + ut_1')\}^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2 \\ &= \beta^2\{(x_2' - x_1') - u\,(t_2 - t_1)\}^2 + (y_2' - y_1')^2 + (z_2' - z_1)^2.\end{aligned}$$

This obviously depends upon u, which enters twice over, once as the multiplier of $t_2' - t_1'$ and again as a component of β, or $1/\sqrt{1 - u^2/c^2}$. It is therefore a new form. The original *form* in the new co-ordinates namely,

$$(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2$$

does not represent the square of PQ, and is therefore not invariant.

But suppose we test

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2$$

instead. We shall now have to use the relation $t_1 = \beta\left(t_1' + \dfrac{ux_1'}{c^2}\right)$ and a corresponding relation for t_2. The expression becomes

$$\beta^2\{(x_2' - x_1') - u(t_2' - t_1')\}^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2$$
$$- c^2\left\{\beta\left(t_2' + \frac{ux_2'}{c^2}\right) - \beta\left(t_1' + \frac{ux_1'}{c^2}\right)\right\}^2.$$

Remembering that $\beta^2 = 1/(1 - u^2/c^2)$, this becomes, after some reduction,

$$(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2 - c^2(t_2' - t_1')^2,$$

which is of exactly the same form as that with which we started, and is independent of u.

With the notation which has been used already, we say that $dx^2 + dy^2 + dz^2 - c^2dt^2$ is invariant. We may continue to call it ds, but ds no longer represents a line. We have, in fact, introduced time into the specification of P and Q. P and Q are therefore what we called events, or point-events, in Chapter III, and ds is not the *distance* between two points, but the *interval* between two events.

We thus see that the use of the Lorentz transformation involves the new term $-c^2dt^2$ in the expression for ds. This is analogous to the introduction of another dimension, with the difference that the new term is subtractive instead of additive. It might be thought that the factor c constituted another point of difference, but since we may choose our units of length and time as we please, we may select them so that c, the velocity of light, is unity. With this understanding the invariant expression is

$$dx^2 + dy^2 + dz^2 - dt^2.$$

If we represent as before any change of reference system using rectangular co-ordinates by dashed letters, we shall have always

$$dx^2 + dy^2 + dz^2 - dt^2 = dx'^2 + dy'^2 + dz'^2 - dt'^2.$$

We have to remember that the use of the Lorentz transformation is not a matter of choice or convenience. It is a necessity if the statements of differently circumstanced observers are to be correlated, and therefore this fourth dimension is forced upon the physicist. He has no option in the matter but to accept the fact that

he has to deal, not with space of three dimensions and an independent time, but with a state of affairs in which all four are inseparably associated. He is obliged to realize that lengths and times as manifested to him are not absolute properties of bodies existing independently of him, but relations between himself and some fundamental entity in which time plays the part of a dimension. We are thus brought back to the point at which we left the relativist in Chapter I, and we are in a position to see what he meant by saying that mathematical processes would distinguish time from length, breadth, and height. The distinction consists in the minus sign prefixed to the time symbol. We also see that the main feature of the statement of physical laws agreeably with the Restricted Principle of Relativity must be the use of a reference system in this unfamiliar compound of space and time. It is only by analogy that the word " space " can be applied to this concept. The concept was arrived at by an application of considerations suggested by the step from two dimensions of space to three, and there is therefore something to be said for retaining the word and extending its meaning. But it is better to use some other word, retaining " space " for its ordinary usage. The word " continuum " suggests itself for reasons which will presently appear.

All attempts to form a picture of a figure in a continuum of four or more dimensions are in the writer's opinion futile. The mathematician is in no difficulty, for he is able to express by means of his formulæ all properties relevant to his purposes without the necessity of forming a picture; a picture would not help

him materially. But this resource is not open to those without mathematical training. Those properties of things which the mathematician can discard as irrelevant are often the very ones upon which others rely for their concepts, and so the plain man is puzzled when he hears the mathematician talk of four dimensions. He does not realize that what the mathematician is thinking of is things which he can put down in a formula, while he himself is thinking of things out of which he can make a picture, and that these are not necessarily the same. It does not occur to him that in the matter of picture-making the mathematician may be in as great difficulties as himself. But though a picture may be just as impossible to the one as to the other, the mathematician has in his formulæ perfectly adequate means of representing, though not of picturing, all he wants.

For example, we have seen that in two dimensional space the expression for ds^2 is $dx^2 + dy^2$, and is invariant. This is an essential property of space of two dimensions, which may therefore be defined as that condition in which this expression is invariant. This is merely a formula, but it is all the mathematician requires. From the mere fact that $dx^2 + dy^2$ is invariant, the mathematician can derive the whole of the geometry of two dimensional space, and it is more or less incidental that in this particular case the expression can be interpreted as the square on a line. Similarly space of three dimensions can be defined as that condition in which the *three*-term expression $dx^2 + dy^2 + dz^2$ is invariant, and again no picture is required. Proceeding in the same way we can say that a four-dimensional

continuum is a condition in which a four-term expression of the same kind is invariant. Now this is perfectly intelligible as far as it goes, and it goes far enough to contain positively *all* that the mathematician wants. Similarly he might proceed to define an n-dimensional continuum in a manner perfectly adequate for his own purposes. He might, of course, have a not unnatural curiosity to know what things would look like in such a continuum, but this is only a matter of mild interest. It is, on the other hand, everything to the plain man, to whom the formula is nothing.

Events whose co-ordinates differ by very little from another are said to be *adjacent*, and it is clear that events may occur so closely in succession, and so near together in space, as to form a series as nearly continuous as we please. Hence the name continuum. The physical history of any object is such a series of events. It is called a world-line. This term is a literal translation of the German "weltlinie". When the world-lines of objects intersect those of observers the objects become manifest as phenomena.

Summary.—The expression for the square of the line element ds in a rectangular reference frame consists of the sum of series of terms $dx^2 + dy^2 + \ldots$ This expression is invariant—that is to say, it suffers no change in magnitude through change of axes of reference. There are as many terms in the expression as there are dimensions in the space under consideration, and therefore, since the Lorentz transformation introduces four terms into it, a four-dimensional continuum is indicated

for the statement of laws conformably to the principle of relativity. Time is distinguished from the other dimensions of this continuum by the sign prefixed to the corresponding term. All geometry can be developed from the bare fact of the invariance of the expression for the square of the line element in terms of the co-ordinates without help from diagrams, and the number of dimensions is no obstacle to mathematical representation. A convenient notation for expressing the line element is explained. Thus, for brevity, $dx^2 + dy^2 + dz^2$ is written instead of

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

when the differences $x_2 - x_1$, etc., are small.

CHAPTER XII

THE GENERAL PRINCIPLE OF RELATIVITY

WE now resume the main subject of the expression of physical laws independently of particular observers where it was left off at the end of Chapter IX. We are about to start on an entirely new inquiry, which, though suggested by what has preceded, has no logical connexion with it. Excepting that the general attitude of mind and way of looking at things, more especially in respect of the four-dimensional continuum, forms a good and practically an indispensable preparation for what follows, logically we might have commenced the whole subject with the general theory, which we are now about to consider. The general theory is not deduced from the restricted theory, though the restricted theory constitutes a particular, or limiting, case of the general theory, of which fact advantage is taken, as will appear in the sequel.* In developing the general theory we are not going to make use of any of the assumptions or postulates relating to the velocity of light, and the inability of an observer to detect his movement in the æther, nor any of the deductions therefrom. In fact, we shall see that the

* Chapter XVIII.

velocity of light is constant only in the absence of a field of gravitational force.

The restricted theory, however, suggests a more general theory in one way. No statement of physical law can be regarded as wholly satisfactory so long as it is confined to unaccelerated systems of reference. The phenomena themselves have no such preference. Physical agencies act whether the regions in which they reside are accelerated or not, and to confine the statements which represent them to unaccelerated systems is an arbitrary restriction which cannot be accepted if it is by any means avoidable. We have, however, seen in Chapter VI the obstacle which the forces and accelerations peculiar to each individual accelerated system present to the adoption of such systems of reference, and it is necessary to add that all systems other than Cartesian suffer from the same disability whether they are subject to a bodily acceleration or not. Polar co-ordinates, and Gaussian co-ordinates generally, which were explained in Chapter III, involve the use of curves in the frame of reference, in part or wholly, and these curves import into the mathematical statements expressions for what are termed centrifugal forces.* It is, unfortunately, not possible with the limited amount of mathematics at our disposal to illustrate this point, and therefore the general reader must be asked to accept the fact that no systems other than Galilean—that is, unaccelerated Cartesian systems—can be used without importing into the formulæ

* The reader who understands Particle Dynamics will see the point at once. It is obvious from the expressions for accelerations in Polar systems.

expressions for forces which are peculiar to each system, and which may therefore be expected to upset anything in the nature of general statements.

The illustrations which we shall be compelled to use will be taken from cases in which the reference systems are subject to bodily acceleration. It will, however, be made clear that these cases are particular instances of change of co-ordinates, and the reader must therefore understand that forces are in general artificially induced by any change of co-ordinates. All reference systems in which curvilinear co-ordinates are used count as accelerated systems, though the propriety of so regarding them may not be as obvious as when bodily acceleration of the system takes place.

There appears, therefore, to be an insuperable obstacle to the statement of physical laws in such a way as to be common to all observers whatever their circumstances, but it may be shown that the difficulty disappears if what is called the "Principle of Equivalence" is granted. It is found that this principle enables us to act upon the postulate that *All Gaussian systems are equally applicable for the statement of general physical laws.* This postulate is the General Principle of Relativity. As Gaussian systems mean practically any reference systems whatever, statements of laws with reference to them will be of the most general character. We shall return to the application of this principle after the argument has been further developed.*

We have seen that in the general case, a change or transformation from one reference system to another

* Chapter XX.

involves the introduction of forces. The principle of equivalence may, with sufficient accuracy for present purposes, be stated as follows: *A gravitational field of force is exactly equivalent to a field of force introduced by a transformation of the co-ordinates of reference, so that by no possible experiment can we distinguish between them.**

It has been shown that in order to carry into effect the mechanical principle of relativity it was necessary to assume that lengths and times were unaltered by relative motion, these suppositions being embodied in the scheme of transformation (1), Chapter VI; also, that the restricted principle of relativity required the supposition that lengths and times altered in a special way with relative motion, which suppositions were embodied in the Lorentz transformation, and this in turn involved the reference of phenomena to a four-dimensional continuum. Now the principle of equivalence stands in the same relation to the general principle of relativity as the suppositions respecting lengths and times stand to the mechanical and restricted principles. It is required in order to carry the general principle into effect. The principle of equivalence, together with the idea of the four-dimensional continuum, are the foundation of the general theory. The last chapter dealt with the four-dimensional continuum; those immediately following will deal with the principle of equivalence.

It will be found that the discussion of the principle of equivalence will disclose the remarkable fact that gravitational forces and the geometrical properties of

* Eddington, "Report on the Relativity Theory of Gravitation," pp. 19, 43; and "Space, Time, and Gravitation," p. 76.

the regions or fields in which these forces occur, are but different aspects of the same thing. This relationship, it will be shown, forms the basis of a new law of gravitation. It will thus be seen that gravitation possesses an importance hitherto unsuspected. Physical agencies, of whatever kind, necessarily conform to the geometry of the region in which they act, and if geometry and gravitation are merely different ways of viewing the same set of facts, it is clear that gravitation likewise controls these agencies. On account of the pre-eminent position assumed by gravitation, the General Theory of Relativity is also called the Gravitational Theory. The remainder of the book will deal with these matters.

Summary.—The restriction of statements of physical law to Galilean systems is arbitrary, but the introduction of new forces is an obstacle to the use of other systems. Any Gaussian system, however, can be used, if advantage is taken of the principle of equivalence. This principle states that gravitational fields and fields of force artificially induced by change of co-ordinates are equivalent. In stating laws in conformity with the general principle, phenomena are regarded as occurring in a four-dimensional continuum. Geometry and gravitation are inter-related.

CHAPTER XIII

ROTATING SYSTEMS

AS the first illustration of the Principle of Equivalence we shall consider the forces induced by rotation and the corresponding geometrical relations.

The parts of a rotating body are subject to accelerations in lines directed towards the centre or axis of rotation. These accelerations arise in accordance with Newton's first law of motion, which states that every body persists in its state of rest or of uniform motion in a straight line unless acted on by a force. The natural tendency of the body is to move along a tangent to the circle which it describes, and this tendency is what is called the *inertia* of the body. Newton's first law is the definition of inertia.

Imagine an observer situated on a platform made to rotate in its own plane with constant angular velocity—that is, equal angles are described in equal times. We shall suppose the platform to be rough, so that the observer can keep his footing, and we shall further suppose the platform to be located somewhere remote from all other objects, and to be so circumstanced otherwise that he has no direct means of perceiving the rotation. If the platform is in the earth's gravitational

field we must suppose it to be horizontal, so that gravity has no moving effect on any of the objects on its surface, but it is better to think of the platform as situated somewhere away in space beyond any gravitational fields due to the presence of extraneous bodies.

The observer will share the rotation of the platform. His inertia will assert itself, and he will therefore be subject to the acceleration to which, as we have seen, all bodies are subject under such conditions, but as he is unaware of the rotation he will not attribute his acceleration to this cause. What he will notice is that as he walks about on the platform he is continually urged away from one particular point, which point is, in fact, the centre of rotation, though he does not recognize it as such. He will also notice that this force acting upon him is exactly proportional to his distance from that point, at which point he finds that it vanishes. His experiments will also show him that the force has the same accelerating effect on all bodies alike. Whatever their mass or material, they will always gain the same outward velocity in the same time, provided only that they are at the same distance from the centre, and that he is careful to remove or allow for all agencies such as friction * or air resistance which might mask the effect of the force upon the body. In accordance with Newton's second law of motion, this equality of

* As it may be asked how the platform could communicate to a body the rotation necessary to set up the centrifugal force if there is no friction, it is suggested that the motion might be communicated by a smooth guide directed towards the point of no force. This would supply the requisite constraint, while allowing the body to move radially.

accelerating effect carries with it the fact that the force acts upon bodies with an intensity proportional to their masses.* If he supposes the seat of the force to be at the central point, and he attempts to screen off the force from the body by the interposition of other objects, he will find that the operation of the force is unaltered. In fact, this force presents the two essential features of gravitation: (1) it has the same *accelerating effect* upon all bodies whatever their mass or constitution, and acts upon them with an *intensity* proportional to the mass; and (2) it cannot be screened off. It is true that it is directed away from a centre instead of towards one, unlike ordinary gravitation, and it acts according to a different law, but this does not affect the main features just mentioned. The observer, in fact, believes himself to be in a gravitational field.

Let us now turn from the effects of forces on the platform to its geometrical relations. Suppose that an airplane flies over it, so that unknown to the observer the path is a straight line relatively to some other observer, who is using an unaccelerated reference frame. What will the path look like to the man on the platform? This will, of course, depend to some extent on his position on the platform, but for simplicity we shall suppose him to stand at the centre. He can identify this point, not by reference to the rotation, for he is ignorant of it, but as the point at which he feels no

* See Chapter IV. If m, m' and F, F' are respectively the masses of two bodies and the forces acting on them, and f the common acceleration, we have by Newton's second law, $F = mf$, and $F' = m'f$; so that $F/F' = m/m'$.

force. The annexed figures show the paths relatively to the two observers.

Fig. 28 shows the path *AB* plotted relatively to an observer *O*, whose position coincides with the centre of rotation of the rotating platform, but who refers the movement of the airplane to an unaccelerated reference frame, *Ox*, *Oy*. This frame will have no rotation since

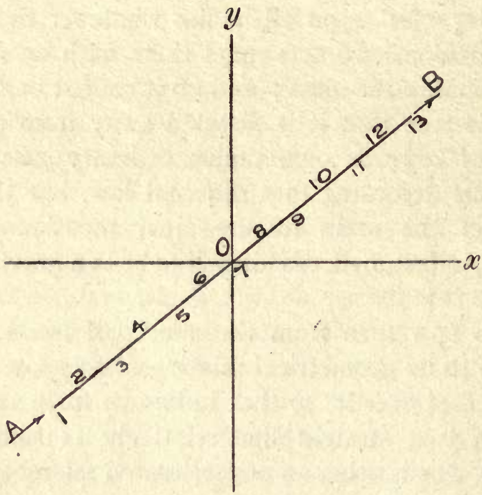

FIG. 28.

it is unaccelerated, and we may consider it fixed. Let *AB*, which passes through a point directly over *O*, be the path of the airplane, and let the numerals 1, 2, etc., represent its successive positions at equal time intervals, say two seconds. It is required to find what appearance the path will present to the observer on the rotating platform, who will refer everything to a reference frame fixed on his platform, as he is unaware

of the rotation. Let this frame be OX, OY. Although O' actually coincides with O, the reference frame of O' is shown in a separate diagram, Fig. 29, for clearness. We will suppose the platform to rotate once in twelve seconds, or 30° in one second. Let the position 1 of the airplane correspond to the time when the two reference frames coincide. We can plot the apparent path

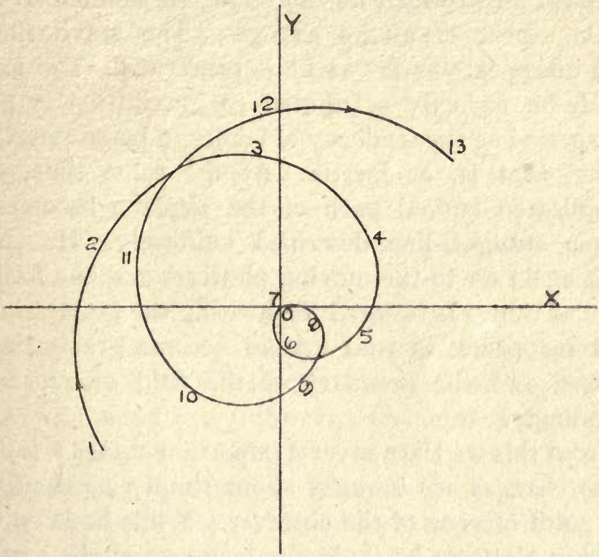

FIG. 29.

on the reference frame OX, OY by pinning two pieces of paper on a drawing-board by a drawing-pin representing the coincident points O, O'. The upper piece of paper carries a diagram of the reference frame Ox, Oy, and the lower one a diagram of OX, OY. We start with both reference frames coincident, and prick through the point 1 in the upper paper so as to make

a corresponding mark in the lower. We then turn the lower paper through 60° relatively to the upper and prick through the point 2 in a similar way, and so on. We remove the upper paper and draw a curve through the marks on the lower, and the curve will represent the path of the airplane as it appears to O'.

We now provide the observer with a non-rotating platform on to which he can step. As he does so the whole aspect of affairs changes. The gravitational field disappears as far as he is concerned. The force which he formerly attributed to gravitation is now interpreted as the tendency of bodies to pursue straight paths—that is, as inertia. At the same time, the complicated curved path of the airplane becomes a simple straight line described uniformly. He steps back again on to the moving platform and the former state of things is restored forthwith ; the gravitational field reappears, inertial masses become gravitational masses, and the geometry of the field alters correspondingly.

From this we learn several important things : (1) All these changes are brought about simply by changing the point of view of the observer. While he is on the moving platform he is, in the language of the mathematician, referring all the circumstances to rotating axes ; while he is on the non-rotating platform he is referring them to fixed axes, and what he does when he steps from one to the other is essentially nothing more than to change over from one set of axes to another, or, as the mathematician would say, the changes are brought about by a transformation of co-ordinates. He has changed from an accelerated (in this instance,

a rotating) system to an unaccelerated one, and the whole set of changes follow accordingly. (2) The physical quantity called mass is interpreted by him in the one case to be gravitational mass, and in the other inertial mass. The two are one and the same thing looked at in different ways. (3) Corresponding to the gravitational field are geometrical relations peculiar to itself. (4) In this particular instance the gravitational field can be completely extinguished by a change of axes. A suitable mathematical transformation transforms the gravitational field out of existence. (5) The gravitational field arises because bodies are guided in a particular way. Bodies rotating with the platform are guided so that they describe circles, with the result that a force arises which has all the essential properties of a gravitational force. While the man is on the rotating disk he looks on the body as acted on by a force at a distance; he looks on a certain point on the disk as a centre of repulsion, as we regard the centre of the earth or of the sun as a centre of attraction. His attention is concentrated on this centre and not on the course of a body. When he is off the disk on the other hand, his attention is concentrated on the circular course, and he ascribes the force, previously thought of as due to some distant agency, to the fact that the body is constrained to move in that particular way.

Summary.—Forces arising on a rotating system from centrifugal action are indistinguishable in principle from gravitational forces. The geometrical relations of the system show a correspondence with the forces.

Mass can be interpreted either as gravitational or inertial, according to the point of view. Whether the mechanical conditions on such a system are to be regarded as gravitational or due to centrifugal action is purely a matter of choice of co-ordinates. A gravitational field artificially induced by rotation can be transformed away completely.

CHAPTER XIV

TRANSLATION

THE last chapter dealt with a case of an accelerated system in which the bodies composing it were subject to an acceleration transverse to their line of motion. The present one will deal with a system subject to an acceleration in the line of motion. The word "acceleration" will therefore approximate in meaning to its popular sense of increase of speed, though it will include decrease. The extension of the use of the word to transverse effects is a refinement which has nothing corresponding to it in popular usage. As has been shown, this extension is justifiable, since the same agency is at work in both cases. All forces produce acceleration in their line of action only, but if a body has already a velocity in some other direction the speed-increasing effect of the force is more or less masked, and may be completely masked, so that acceleration may, as we have already seen, show itself in a bending of the path of the body only.

The subject matter of the present chapter is dealt with in Chapter XX of Einstein's book in a way which leaves no room for improvement in clearness or simplicity. What we have to say, therefore, can only be for the most part paraphrase.

Let us imagine ourselves in a closed box like a room situated in some region of space where there is no gravitational field. There will be no such thing as weight. If we put an object in mid-air and leave it, it will remain where we put it, but the slightest touch will send it moving off uniformly in a straight line in the direction in which it was pushed, with greater or less speed, according to the strength of the touch. When the object encounters a boundary of the room (we cannot now say wall, or floor, or ceiling, for there is no up or down or sideways) it stays there, if it is not elastic. If it is perfectly elastic it will rebound with the same velocity and continue in perpetual movement. When we push the object we ourselves will recoil and continue to recoil, until we are brought up by a boundary. Indeed, the least push against a side of the box or against any object on our part will set us going—perhaps through the air—so that if we want to stay in any particular place we must tie ourselves there.

But now suppose that a rope is hooked on to the box outside, and some being, no matter how, pulls on the rope so as to give the box a uniform acceleration. Immediately everything in the box which is not already at the side remote from the rope attachment gets left behind as the box moves forward, and from the inside things have the appearance of falling towards this side—now the floor—with uniform acceleration. All objects within the box which are not already on the "floor" are affected alike. If there is no air in the box they all fall with the same acceleration, just as in a gravitational field. If there is air it is carried along with the

box, and partly carries the objects with it to a greater or less degree, according to their density, size, and shape, thus exactly imitating the effect of air resistance on falling bodies. Anyone standing on the floor when the motion started will immediately feel the sensation of weight, and will have to support himself by his legs. If he tries to prevent things from reaching the floor—falling, as he thinks—he will find that the things which are hardest to support are those which were hardest to move before the box began to accelerate. If he is at a loss to account for these phenomena, and he chances to look towards what is now the ceiling, he may see part of the hook attachment, and he may very well conclude that he is suspended by it in a gravitational field ; it is quite likely that it may never strike him that the whole thing is simply the result of the box having been set in motion with an ever-quickening velocity. Objects which he thinks are falling are simply being left behind as the box moves. Again, if he attaches a body to the ceiling by a string, the string will be put in tension, due, as the observer inside thinks, to the weight of the body, but, as the being outside thinks, to the fact that the box is pulling the body along with it.

Accompanying all this there is a corresponding distortion of the paths of moving objects. Bodies thrown across the box which formerly described straight lines, now describe parabolas, or, if air resistance is taken into account, trajectories exactly like those in a gravitational field.

In short, we get a set of circumstances exactly parallel to those examined in the last chapter. There

is no distinguishable difference between the phenomena inside the box whether they are regarded as due to a gravitational field or to an acceleration impressed on the box. To a person inside they have the appearance of occurring in a gravitational field ; to a person outside they are due to inertia. To both, inertial and gravitational mass are the same. Which way we regard the phenomena is indifferent ; it is all a question of point of view, or, to put it mathematically, of choice of axes. The observer inside the box refers phenomena to a reference frame moving with the box, and fixed relatively to himself ; the observer outside refers them to a frame fixed with reference to *himself*.

Summary.—Systems in which acceleration takes the form of change of speed only, exhibit the same features as those described in the last chapter with reference to rotating systems. The phenomena can be interpreted as due either to acceleration or to an artificial gravitational field. The gravitational field can be wholly transformed away by a change of axes, and a change of axes from one point of view to the other is accompanied by a parallel change in geometry.

CHAPTER XV

NATURAL GRAVITATIONAL FIELDS

A CHARACTERISTIC feature of the instances considered in the last two chapters is that the gravitational field can be completely destroyed, as such, by a suitable transformation to another frame of reference. All forces attributed to gravitation before the transformation are attributed to inertia after it, and this effect extends to the whole of the field. We have now to inquire whether gravitational fields such as occur in nature—the earth's gravitational field, for example—can be transformed out of existence by a similar process.

Imagine an observer enclosed in a box as before, but falling freely in the earth's field. An observer on earth, who regards the occurrence from the point of view of a fixed earth—that is, he refers it to a fixed frame of reference—is conscious of the earth's gravitational force, and he attributes the motion of the box to that cause. But an observer inside the box is not conscious of any force whatever. The acceleration acts equally on the box itself and all the things inside it, including the observer himself. A body placed in mid-air in the box will remain in the same position relatively to the box, and the observer will feel no weight. If he wishes

to remain in one position he has to fasten himself there; and a body thrown from one side to the other will describe apparently a straight line. As far as the interior of the box is concerned, matters are exactly as described in the last chapter before the box, which was then considered, had been given an acceleration.

But now suppose the size of the box to increase, as shown in Fig. 30, so that the slope of radii drawn to

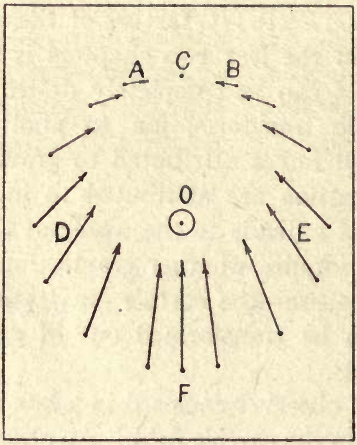

FIG. 30.

the centre of the earth *O* from places near the sides of the box becomes appreciable. Objects at places such as *A* or *B* would actually fall, from the point of view of an observer on earth, along radii *AO*, or *BO*, but from the point of view of an observer situated at *C* inside the box, they would appear to move with an acceleration directed inwards towards him with a slightly upward tendency, as shown by the short arrows, while an object at *C*, where the observer is situated, would remain

apparently in mid-air. If the box were large enough to include the whole earth so that objects at *D*, *E*, and *F* could be observed, it would be seen that the inward tendency increased as far as *D* and *E* and then diminished, while the upward tendency—that is, the tendency towards the observer at *C*—increased all the way round to *F*, where it would be exactly double of the *actual* downward tendency of the observer towards the earth.* The arrows show the approximate relative magnitudes and directions of these tendencies as they appear to the observer in the box.

It seems then that though it is possible by using a suitable system of reference to transform away gravitation in a small region—which may be anywhere, since we have not specified any particular position for the observer—this transformation, so far from annihilating the whole field, only aggravates the effects of gravitation in the remainder. Gravitation can, so to speak, be smoothed out in one place only to appear with greater intensity in another.† But still we have the fact that anyone can, in his own neighbourhood, produce all the effects considered in the last two chapters. By merely altering his point of view an observer can, in any small region, regard a force as either inertial or gravitational, and, as before, his geometry will follow his choice.

In the last two chapters no stipulation was made as

* Readers familiar with the parallelogram of accelerations can easily verify these statements.

† Newton uses exactly this transformation in that famous proposition LXVI, Book I, of the "Principia". See also Herschel's "Outlines of Astronomy," 1878 Edition, § 610, p. 415; also Proctor, "Old and New Astronomy," pp. 207, 208.

to the size either of the revolving platform or of the observer's box; the transformations there considered included all space. We therefore have to distinguish between the cases in which gravitation can be annihilated everywhere, and those in which it can be annihilated throughout a small region only at a time. In the former case the gravitation is wholly attributable to choice of axes of reference; in the latter it is due to the presence of attracting matter. We may distinguish the two by calling the former an *artificial* gravitational field and the latter a *natural* one.

These facts lead to the following re-statement of the principle of equivalence, which includes both cases:—

*A gravitational field of force is precisely equivalent to an artificial one, so that in any small region it is impossible by any conceivable experiment to distinguish between them.**

The limitation to small regions does not exclude cases where the whole gravitational field is artificial and can be extinguished by one transformation. For if all small regions happened to be alike, a transformation applied to one would affect all equally.

We have seen that the fields of force which have been considered carry with them their own peculiar geometry. The principle of equivalence, therefore, involves a relation between gravitation and geometry and suggests the general possibility of a relation between the gravitational forces in any region and the geometry of that region, so that the specification of the one carries with it the specification of the other. Now a relation of this kind can be nothing else than a state-

* Eddington, "Space, Time, and Gravitation," p. 76.

ment of gravitational forces in terms of geometry ; it is, in fact, a Law of Gravitation, and it is to the possibility of obtaining such a law that the principle of equivalence directly points.

An important question, however, arises. It is very well to say that if we can specify either the geometry or the gravitational forces we can specify the other, but how can we specify either when they are both unknown ? And what in particular is meant by specifying geometry ? In the next chapter a method of specifying geometry will be examined which will answer this latter question and allow us to proceed with the answer to the former. It will afterwards be seen that this method directly suggests an hypothesis connecting gravitation and geometry, which hypothesis is Einstein's Law of Gravitation.

There is one possible difficulty with which it is desirable to deal before proceeding further. It may occur to the reader to ask what is the use of knowing either the geometry or the forces if our knowledge applies to a small region of the field only. It may not be clear to him how a transformation applied to one part only of the field can affect the whole. The answer to this question is that the properties, metrical and otherwise, of any region are taken as continuous—that is, there are no sudden or capricious jumps between the properties of any particular region and the next. The geometrical properties which it is our object to specify will include the rate of variation of the various quantities from place to place, and therefore if we know the properties in one place we have the means, theoretically at least, of knowing them everywhere, much in the

same way as a man who knows how much property he has at the present, and how fast it is increasing or dwindling, has the means of forecasting the probable state of his affairs in the future. If, therefore, there is any place where we cannot specify either the gravitational forces, or their rates of change, we cannot successfully apply at that place any transformation with the object of destroying the gravitational field. Such a condition arises where attracting matter exists which is creating the gravitational field. We therefore exclude such places from the operation of the principle of equivalence. It will be seen that this point is not without importance in the sequel.

Summary.—A natural gravitational field can be extinguished locally by a suitable change of point of view, but this local extinction modifies gravitational effects elsewhere. The principle of equivalence can be stated in terms appropriate to a natural gravitational field, and it indicates a relation between geometry and gravitation. A method of specifying geometry is required. A natural gravitational field cannot be transformed away where attracting matter exists.

CHAPTER XVI

GEOMETRY OF THE GRAVITATION THEORY

IN order to describe the metrical properties of any region mathematicians have resorted to several methods, of which the system of Euclid is an example. The method, however, which now concerns us is based on the forms which the expression for the line element *ds* assumes under different conditions. In this chapter we shall confine our attention to space of two dimensions. The sense in which the word "dimensions" is used must not be confused with its popular meaning of magnitude. In the present chapter, and for the most part elsewhere in this book, the word has reference to the number of independent quantities which are required to define a point or point-event. Thus, in the space of experience, three independent co-ordinates are required, and so we call it three-dimensional space. When we speak of the three dimensions of space we are simply referring in general terms to the measurements in three different directions necessary to locate a position. On a surface, plane or otherwise, two co-ordinates only are required when the surface has been decided upon, and we express this fact by saying that we are in, or are considering, two-dimensional space. The idea of *curved* two-dimensional space, however, presents great difficulty to many who readily accept the term two-dimensional space as an intelligible

description of a plane. They can understand the suppression of the third dimension in "flat-land," but not, for example, on the surface of a sphere. The curvature of the sphere seems to obtrude itself, and force the third dimension on the attention. As a matter of fact, three dimensions are just as necessary to the appreciation of flatness as to that of curvature. Flatness is unmeaning without the corresponding idea of something which is not flat, and in the absence of the faculty of appreciating a third dimension, no comparison can be made. A "flat-land" being who has no such faculty, transferred from a plane to the surface of a sphere, would have no direct means of perceiving any change, though he might, theoretically at least, employ the indirect means described at the end of the present chapter. These means, however, do not help him to visualize a third dimension, nor do they alter the fact that positions in his space require two co-ordinates only to locate them.

It was seen in Chapter XI that when plane rectangular reference systems are used any short line ds in two dimensions can be expressed by the relation $ds^2 = dx^2 + dy^2$.* This relation, however, holds good only for rectangular systems. If other systems are used, such as polar or Gaussian co-ordinates, the expression for the line element becomes more elaborate. It is beyond the scope of this book to give this expression

* This relation also holds good for cones and cylinders, which can be formed by curling a plane, *e.g.*, rolling up a flat piece of paper. This refinement, however, is not required for present purposes. When curved surfaces are mentioned in the text it will be understood that reference is made to surfaces, such as that of a sphere, which cannot be flattened.

in its most general form,* but it may be considered to be sufficiently exemplified for two dimensions by the relation

$$ds^2 = g_1 dx_1^2 + g_2 dx_2^2 \quad . \quad . \quad . \quad (8)$$

which is a particular case of it. It will be observed that the Cartesian relation $ds^2 = dx^2 + dy^2$ is the particular form which this standard relation assumes when g_1 and g_2 are each equal to unity, and x_1 and x_2 are identified respectively with x and y. The present chapter will consist of an examination of the meaning of the multipliers g_1 and g_2 when x_1 and x_2 signify other kinds of co-ordinates, such as the radius vector and vectorial angle † of polar co-ordinates, or latitude and longitude on a sphere. The immediate point which the following examples illustrate is a very simple one, namely, that the values of the g's in the relation (8) depend upon the system of reference used, and upon the curvature of the surface upon which the line element is drawn. This might, indeed, be taken as obvious, but the illustrations lead to some further considerations which are necessary to the development of the subject.

Plane Polar Co-ordinates ‡

Let the point P (Fig. 31) be located by the polar co-ordinates (r, θ) as explained in Chapter III, and let PQ be a line element extending from P to a point Q, whose co-ordinates are determined by adding on any

* Eddington, "Space, Time, and Gravitation," p. 82.

† *See* Chapter III for definitions of these terms.

‡ The reader will remember, from Chapter III, that these are nothing more than range and bearing, the radius vector, r, being the range, and the vectorial angle, θ, the bearing.

small quantity dr to r and a corresponding small quantity $d\theta$ to θ, as shown in the figure. As we are considering small quantities only, it does not matter whether the element PQ forms part of a curve or not; to all intents and purposes it is straight. Draw PR perpendicular to OQ. Now the smaller we take $d\theta$ to

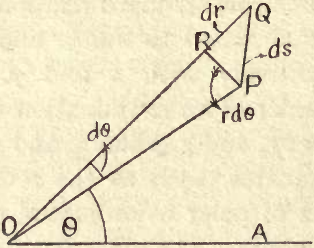

FIG. 31.

be, the closer together are P and R, so that as $d\theta$ is very small, we may consider OP, or r, to be equal to OR. Thus $OR = r$, and therefore $RQ = dr$. For the same reason we may suppose PR to be equal to a small circular arc struck with O as centre and OP as radius, and therefore the angle ROP or $d\theta$ is RP/r radians.*

* The reader who is unacquainted with the "circular measure" of angles may be informed that, according to this system, the unit of angular measurement is the angle POQ (Fig. 32), subtended at the centre, O, of a circle by an arc, PQ, equal in length to the radius, OQ. This angle is called a "radian". It is about 57 degrees. Any angle is measured by its ratio to this unit, and is therefore equal to the ratio of the

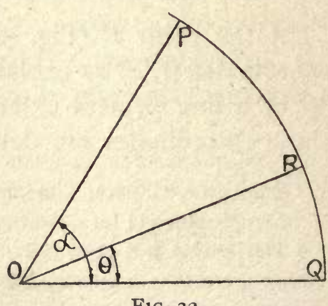

FIG. 32.

Thus

$$RP/r = d\theta, \text{ or } RP = rd\theta.$$

But, since PQR is a right-angled triangle, we have

$$PQ^2 = QR^2 + RP^2,$$

or

$$ds^2 = dr^2 + r^2 d\theta^2.$$

Thus, if we identify r with x_1, and θ with x_2 in the standard relation, we see that $g_1 = 1$, and $g_2 = r^2$.

We now notice a point which is most important for our purposes, and upon which due stress will be laid in the sequel. If we had chosen to refer the positions of P and Q to rectangular co-ordinates, we could, as has been seen, have expressed ds^2 in the form $dx_1^2 + dx_2^2$ by identifying x_1 and x_2 with the Cartesian co-ordinates (x, y) of P. The transformation—or change of point of view, as we have expressed it—from a polar frame of reference to a Cartesian frame has the effect of replacing r^2 by unity. This is a general characteristic of plane geometry. In plane geometry every system of reference has its own peculiar corresponding expression for the line element, but it is always possible by changing over to a rectangular Cartesian system to transform this expression for the line element into the form $dx_1^2 + dx_2^2$, thus reducing both g_1 and g_2 to unity.*

arc which subtends it to the radius. Thus, if ROQ or θ be such an angle and a a radian, we have from the annexed figure, $\frac{\theta}{a} = \frac{RQ}{PQ} = \frac{RQ}{OQ} \div \frac{PQ}{OQ}$. But $PQ = OQ$, and $a = 1$, by definition, and therefore $\theta = \frac{RQ}{OQ}$.

* We omit to refer explicitly in the text to the more complicated cases where the product $dxdy$ occurs in the line element. It disappears on transformation to a rectangular system.

Longitude and Latitude

Consider the small triangle QPR drawn on a sphere, one-eighth part of which is shown in Fig. 32. The sides of the triangle, of which PQ, or ds, is the hypotenuse, are now necessarily curved, but this need not

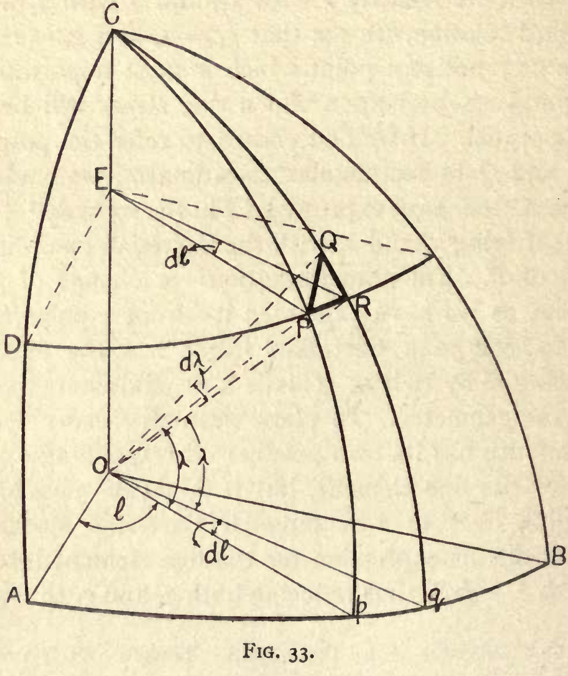

Fig. 33.

trouble us, for, as before, the triangle is taken so small that the sides are substantially straight. We may take the earth as a specimen of a sphere, and speak throughout in geographical terms. Let AB be the equator and C the North Pole. AC is the zero meridian of longitude —let us say the meridian of Greenwich. We will take

P in north latitude λ and east longitude l. Thus, if we draw the meridian CPp through P, and join O, the centre of the earth, to P and p, we have $AOp = l$, and $POp = \lambda$. Take some point Q near P not on the same parallel of latitude, or meridian of longitude, draw the parallel of latitude DPR through P, and the meridian of longitude CQq through Q, and let them meet in R. Let the centre of the parallel of latitude DPR be at E, which will be on the earth's polar axis OC. Let dl and $d\lambda$ be respectively the difference of longitude and latitude of P and Q, so that if we draw the remaining lines shown in the figure we have

$$\angle AOp = l,\ \angle AOq = l + dl,\ \angle pOq = \angle PER = dl\,;$$
$$\angle POp = \angle ROq = \lambda,\ \angle QOq = \lambda + d\lambda,\ \angle QOR = d\lambda.$$

Let $PE = p$, and let OR, the radius of the earth, be a. Then by the rule of circular measure already explained,

$$\angle PER \text{ (or } dl) = \frac{PR}{EP} = PR/p.$$

Thus $$PR = p\,dl.$$

Also, $$\angle QOR \text{ (or } d\lambda) = \frac{QR}{OR} = QR/a.$$

Thus $$QR = a\,d\lambda.$$

Then, since $\angle QRP$ is a right angle,

$$PQ^2 = PR^2 + QR^2,$$

or $$ds^2 = p^2dl^2 + a^2d\lambda^2,$$

if we call PQ, ds, according to previous practice. Whence, if we identify l with x_1 and λ with x_2 in the relation (8) we see that $g_1 = p^2$, and $g_2 = a^2$.

The length p depends upon the radius of the sphere, and also upon the latitude λ*. It obviously depends

* Readers acquainted with trigonometry will see that $p = a \cos \lambda$.

upon the size of the sphere—that is, upon the radius—and therefore upon the curvature. It also depends upon the co-ordinate λ, since if P were at the pole its magnitude would be zero, and if P were on the equator it would be equal to a. Thus, taken together, the multipliers g depend upon the system of reference chosen and also upon the curvature of the sphere.

The same argument might be applied to line elements drawn on other surfaces, say the surface of an egg or a rugby football. We should in such cases get still more complicated expressions for the line element, but they would all illustrate the general fact that the g's in the standard expression for the line element depend upon the co-ordinates chosen and upon the curvature of the surface upon which the element is drawn. The same is true with reference to the plane, but in the case of the plane the entry of the curvature is not so obvious, since it is zero.

It was noticed that in the geometry of the plane the standard expression for the line element could always be reduced to the form $dx_1^2 + dx_2^2$ by a suitable transformation, the g's becoming equal to unity. This is not the case with the sphere or with any other curved surface.* The impossibility of reducing the g's to unity is the necessary consequence of the impossibility of applying a curved surface to a plane so that they shall fit together without distortion. This transformation can, however, be effected at any one spot at a time for a small region on the surface. Suppose, for instance, that AB (Fig. 34) is the trace on the paper of a plane

* Excepting cones and cylinders. *See* previous note.

which touches any curved surface at P. It is clear that a small figure, say a triangle, drawn on the surface at P could be projected on to the tangent plane without substantial distortion. To all intents and purposes it might as well be drawn on the tangent plane. This being so, plane geometry can be applied to it, and the g's can be transformed away. But a figure drawn at Q some distance away from P could not be projected on to the tangent plane at P without considerable distortion. A transformation which would reduce the g's to unity might of course be applied at Q, but this would only produce distortion at P.

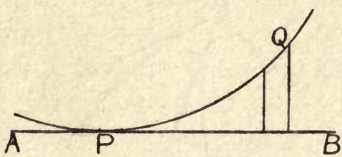

FIG. 34.

There is one important case in which this local transformation cannot be applied. It is necessary for its success that the curvature at the place of application should be continuous. If there is any sudden change, such as would be produced by a sharp ridge or elevation, the transformation could not be effected. Consider, for example, Fig. 35. If two parts PQ, PR of the surface meet at P, making a finite angle TPT', a figure near P on the part PQ would project without distortion on to the tangent plane PT, but a corresponding figure on the part PR would not. So also a figure near P on the part PR would project on to PT' without distortion, but not on to PT. If we take any other plane PS

through P as the plane of projection, figures on neither part could be projected without distortion. It is, therefore, impossible to choose any frame of reference at such points as P in the neighbourhood of which the quantities g may be reduced to unity. It may be done, of course, for other points, but we are no longer able to say that it can be done at *any* point. This is only possible if the surface is continuously curved throughout—that is to say, no ridges, such as that at P, may occur anywhere.

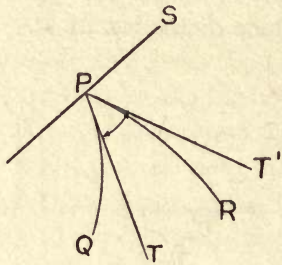

FIG. 35.

It is not necessary to the truth of the relation (8) in any of its forms that the line elements, or the small triangles related thereto, should be drawn on actual physically existing surfaces. The relation

$$ds^2 = g_1 dx_1^2 + g_2 dx_2^2,$$

where g_1 and g_2 are given quantities involving x_1 and x_2, is a condition which could be complied with if the surface on which the elementary figure is drawn were removed, leaving the figure itself in the air, but in the same position relative to the reference frame. The relation defines the kind of surface on to which a line

element would fit if the surface actually existed. Looked at in this way the relation defines, not so much a surface, as the metrical properties of the region of space in which the relation obtains. It imparts a structure or curvature, as it were, to space which limits complete freedom of movement, and it has been found that all the g's appropriate to a particular conformation of space satisfy a corresponding set of conditions, no matter what admissible reference frame may be used.* Thus all the g's corresponding to plane geometry comply with the same set of conditions, whether the reference frame be rectangular, polar, or any other which can be used on a plane surface, those corresponding to spherical geometry comply with another set, and so on. The g's thus furnish a basis on which the geometry of any region can be worked out, that is to say, they specify the geometry. A being inhabiting two-dimensional space, and incapable of perceiving a third dimension, could nevertheless determine the curvature of his space though he cannot visualize the curvature. For he could measure up a number of triangles, and by comparing the results determine empirically, in theory at least, the g's in the relation

$$ds^2 = g_1 dx_1^2 + g_2 dx_2^2.$$

Summary.—The number of dimensions of space is the number of co-ordinates necessary to determine a point in it. The geometrical properties of a region are specified by the form assumed by the expression for

* Eddington, "Space, Time, and Gravitation," Chapter V.

the line element. For two dimensions of space the typical form is taken as $g_1dx_1^2 + g_2dx_2^2$. It is shown by examples that the g's depend upon the reference frame chosen and upon the curvature of the surface on which the line element is situated. If it is on a plane the g's may be reduced to unity by a suitable transformation, but not if it is on a curved surface. This reduction may, however, be effected locally on any surface excepting where the curvature is discontinuous. Local transformation produces distortion elsewhere. The expression for the line element determines the metrical properties of the region in which the element occurs, though no surface on which it might be fitted actually exists physically, and the geometry of the region is thus seen to be specified by the g's. Theoretically, the curvature of a space could be determined empirically without the need for visualizing it.

CHAPTER XVII

GEOMETRY OF THE GRAVITATION THEORY—
(*continued*)

THE results of the last chapter for space of two dimensions have their counterpart in space of three or more, but since this extension carries us beyond our visualizing powers it is necessary to inquire what means exist for representing these results, and in what sense the same terminology can be used.

It was seen in the last chapter that in two-dimensional space a point is located by two independent quantities only, namely, its co-ordinates. This kind of space is therefore necessarily a surface, for we can proceed along a surface in any two independent directions we please, but we should have to leave the surface in order to proceed in a third independent direction. Every surface, whether plane or curved, therefore constitutes a two-dimensional region, and we may name it accordingly. Thus planes, or spheres, or other curved surfaces are respectively plane, spherical, or curved two-dimensional regions. If the standard relation is such that figures can only be drawn which would fit on a curved surface, we say that we are in two-dimensional space curved in three dimensions. It is usual to call plane space *Euclidian* space, because, as has been seen, the theorem of Pythagoras, which is perhaps the most important and characteristic of Euclid's propositions,

holds good with reference to it. This theorem does not hold good on a sphere or other curved surface, and curved spaces are therefore classed as non-Euclidian.

In the last chapter the line element in two dimensions was expressed by the standard relation

$$ds^2 = g_1 dx_1^2 + g_2 dx_2^2,$$

and the matter we have now to examine is the meaning to be attached to a corresponding relation if we proceed to three dimensions, thereby adding another term, which converts the relation into

$$ds^2 = g_1 dx_1^2 + g_2 dx_2^2 + g_3 dx_3^2.$$

If g_1, g_2, and g_3 are all unity, or if they can be made unity by any change of reference system, the relation becomes

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2,$$

which is the ordinary three-dimensional form of the theorem of Pythagoras. No question of curvature arises any more than in the corresponding two-dimensional case. We may therefore carry forward the same terminology as before and say that we are in Euclidian space of three-dimensions, and we can visualize the whole circumstances. But if the g's cannot be made independent of the co-ordinates, we have a situation parallel to the two-dimensional case, in which we had to resort to a curved surface in three-dimensional space on which to draw our two-dimensional figures—two-dimensional, as has been seen, because defined with reference to two co-ordinates only. But in the present case we are in three dimensions already, and it would therefore seem necessary to describe the property imported into the expression by the extra term as

curvature in space of four dimensions. This is obviously beyond our powers of visualizing, and we have to seek for a representation by a method corresponding to that which was applied in Chapter XI. We therefore recur to the case of two dimensions, and ask what picture was presented to us when a plane was, as it were, distorted into a surface by the introduction of appropriate values of the *g*'s into the expression for the line element, and what is the mathematical expression of this picture.

It was seen in the last chapter, or, rather, it was asserted—for the mathematical proof is beyond the scope of this book—that an expression could be found for the curvature in terms of the *g*'s and any reference system which could be used.* We are thus presented with alternatives. We may either picture the curvature of a surface in three dimensions as its defect from flatness, or we may define it by the mathematical formula which represents this defect. We cannot generalize the former, for pictures cannot be made in four dimensions, but there is no difficulty at all about generalizing the latter, beyond some additional complication. We therefore define the curvature of a surface by its mathematical expression, and generalize this expression according to the number of independent co-ordinates—that is to say, dimensions—which are being used. Thus if the *g*'s in the relation

$$ds^2 = g_1 dx_1^2 + g_2 dx_2^2 + g_3 dx_3^2$$

cannot all be made independent of the co-ordinates, we

* Mathematical readers will of course recognize that allusion is here made to Gauss' expression for curvature.

say that we are in three-dimensional space curved in four dimensions, and by *curvature* this generalized expression is meant.

We may proceed in the same way for four or any number of dimensions. The present subject is limited to four. The standard relation may be written

$$ds^2 = g_1dx_1^2 + g_2dx_2^2 + g_3dx_3^2 + g_4dx_4^2 \text{*} \quad . \quad (9)$$

and we say that if any transformation will reduce all the g's to unity we are in Euclidian space of four dimensions; or if any of them cannot be expressed except as depending on the co-ordinates, then we are in space of four dimensions curved in a fifth.

Mathematically expressible conditions exist for the properties of continuity and discontinuity of curvature, which were considered in the last chapter. These expressions are perfectly adequate representations of the properties, though, of course, they are in no sense pictures. With this understanding such statements as that four-dimensional curved space can be reduced to four-dimensional Euclidian space, excepting at points of discontinuity, is perfectly intelligible when stated mathematically. It means simply that when the expression for the curvature complies with the condition of continuity throughout any small region, the g's can be reduced to unity. As an illustration of the procedure we conclude the chapter with an explanation of the generalized meaning of the term "small region" which has been used above.

* The reader may perhaps be reminded that this is not the most general expression for ds^2. Ten terms are actually required, the remaining six containing products such as dx_1dx_2. The expression in the text is taken as a standard for illustrative purposes only.

To arrive at this meaning we ask what is the characteristic mathematical feature of a small region of two or three dimensions. Simply this, that the co-ordinates of all points within it differ from one another by very little. A small region in any number of dimensions is, therefore, an aggregate of points whose co-ordinates differ by very little from one another. But a "point" in space of more than three dimensions has not been defined. In order to do this we notice that a point in two or three dimensions is determined by its co-ordinates (x_1, x_2) or (x_1, x_2, x_3). Its mathematical definition is a set of quantities (x_1, x_2, x_3) taken in that order.* For any number of dimensions, therefore, a point is such a set of quantities as $(x_1, x_2 \ldots x_n)$ taken in that order. A small n-dimensional region is thus an aggregate of such sets where all the x_1's, x_2's, . . . x_n's are nearly equal.

Summary.—Two-dimensional space is a region in which two independent co-ordinates only are required to define a point. It is necessarily superficial, but the surface may be either flat or curved. Space in which all the g's can be reduced to unity is called Euclidian space, whatever the number of dimensions. To generalize the notion of two-dimensional space curved in a third, curvature is defined by a certain mathematical expression, which is then generalized. The ideas of continuous and discontinuous curvature may be represented in like manner by mathematical symbols, however many dimensions there may be under consideration.

* Professor G. B. Mathews, F.R.S., "Nature," Vol. 106, February 17, 1921, p. 290.

CHAPTER XVIII

THE GRAVITATION THEORY

THE results of the last five chapters will now be set down in parallel columns :—

GRAVITATIONAL FIELDS.	GEOMETRY.
(1) An artificial gravitational field can be destroyed, in other words transformed away, by changing the system of reference appropriately.	(1) In Euclidian space the *g*'s in the expression for a line element can be reduced to unity by a suitable transformation.
(2) A natural gravitational field cannot be transformed away wholly.	(2) In non-Euclidian, or curved, space no transformation exists which will make all the *g*'s unity everywhere.
(3) A natural gravitational field can be transformed away locally.	(3) Non-Euclidian space, if of continuous curvature, can be reduced locally to Euclidian space, the *g*'s becoming unity for a limited region.
(4) A local transformation distorts a natural gravitational field elsewhere.	(4) Local reduction to Euclidian space distorts non-Euclidian space elsewhere.
(5) A natural gravitational field cannot be transformed away where matter exists.	(5) Non-Euclidian space cannot be reduced to Euclidian space where discontinuous curvature occurs.

As far as the illustrations in the previous chapters go, geometry and gravitation thus run on parallel lines,

and this suggests that they may always do so. The illustrations raise the presumption that the g's which specify the geometry of a region also specify the gravitational forces in that region, whether natural or artificial, so that the five points set out above are not merely parallel but connected. The supposition includes, for example, the presumption that where the g's can be reduced to unity no gravitational forces other than artificial exist, and if the g's are reduced to unity by transformation, the same transformation *ipso facto* destroys the gravitational field: gravitational forces, if there are any, are necessarily artificial in a Euclidian region. In fact, geometry, as expressed by the g's is the exact counterpart of gravitation, and geometry and gravitation are but different aspects of the same thing.

Stated in this way the supposition is too vague to enable any deductions to be drawn from it. In order to obtain results which can be tested, it must be put into definite mathematical shape—that is to say, embodied in one or more equations, and this is where the difficult and advanced mathematical work comes in. Einstein embodied his theory in a set of six equations, but it is not possible to give the work by which he obtained them, or even to state them. Little more can be done than to state the problem and the result.

We carry forward from the restricted theory the notion of a four-dimensional continuum, and we have, therefore, from these equations to determine in terms of the co-ordinates the four g's in the relation

$$ds^2 = g_1 dx_1^2 + g_2 dx_2^2 + g_3 dx_3^2 + g_4 dx_4^2.$$

Here again the restricted theory helps by supplying the form which the expression for the interval element assumes for unaccelerated systems—that is, for cases where there are no gravitational forces. This is the case in regions of a gravitational field so remote that the attracting matter which produces the field has no effect. Whatever values therefore are found for the g's they must be such as will reduce to the values -1, -1, -1, and $+1$ * at great distances from the attracting matter. The expression which Einstein obtains for the interval element, as the result of solving the six equations, is

$$ds^2 = -\frac{dr^2}{\gamma} - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 + \gamma dt^2 \dagger \quad . \quad (10)$$

where r, θ and ϕ correspond to the polar co-ordinates explained in Chapter III (2), and γ is written for brevity for $1 - \frac{2m}{r}$. m is the mass of the attractive particle to which the field is due. When r is very great, as it is in remote parts of the field, $1 - \frac{2m}{r}$, or γ,

* It will probably be noticed that we have changed signs in the expression for ds^2 given in Chapter XI, writing $ds^2 = -dx^2 - dy^2 - dz^2 + dt^2$ instead of $ds^2 = dx^2 + dy^2 + dz^2 - dt^2$. We have thus written $-ds^2$ for ds^2. The sign given to ds^2 is a matter of convention, and the first of the forms given above is preferable, having regard to the fact that dt^2 is usually much larger than dx^2, dy^2 or dz^2, as will be seen later on in the chapter. The present convention therefore keeps ds^2 essentially positive.

† The reader who is unacquainted with trigonometry need not take any notice of the symbol $sin^2\theta$. In the subsequent work we shall adopt a simplification which will suppress it. It is desirable in the present instance to give the complete formula.

approaches unity.* If we put $\gamma = 1$ in the relation (10) we get

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + dt^2,$$

which, by a transformation which the reader may take for granted, can be shown to be the same thing as

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2,$$

in agreement with the restricted theory.

If r, θ, ϕ, and t be identified respectively with x_1, x_2, x_3, and x_4 in the standard expression for the interval element, we see that

$$g_1 = -\frac{1}{1 - \frac{2m}{r}} \qquad g_3 = -r^2 \sin^2\theta$$

$$g_2 = -r^2 \qquad g_4 = 1 - \frac{2m}{r}.$$

The reader does not need to be reminded that all this mathematical work is simply the statement of an hypothesis. The relation (10) results from the solution of a certain set of six equations which must be taken as a particular mathematical embodiment of the general supposition that the g's which determine the geometry of a region also determine the gravitational forces. These equations, which it has not been possible to give on account of their complexity, constitute the hypothetical law of gravitation which Einstein puts forward. He determines by means of them the values for the g's which have just been given, and he says that the g's

* The same result follows if no matter is present anywhere, for then $m = 0$, and $\gamma = 1$ everywhere.

represent both the physical and the geometrical state of the gravitational field. This is nothing more than a plausible conjecture until it has passed the test of experiment, like every other hypothesis at the same stage. The experimental tests will be given in the next chapter. Meanwhile we proceed to make some further observations on the formula (10) in the simplified form

$$ds^2 = -\frac{dr^2}{1 - \dfrac{2m}{r}} - r^2 d\theta^2 + \left(1 - \frac{2m}{r}\right)dt^2 \quad . \quad (11)$$

obtained by suppressing the space co-ordinate ϕ, and thus reducing the conditions to two dimensions in space and one dimension in time.

No transformation will reduce the g's in this expression for the interval element to unity, consequently space in a gravitational field is non-Euclidian. It has a twist or curvature, and no figure obeying the theorem of Pythagoras can be drawn in it. We infer, therefore, that in our actual physical conditions Euclid's system is not exactly true—that is to say, it does not exactly correspond to physical measurements. It might be thought that possibly this is due to the fact that the time dt (or dx_4) comes into interval measurements. This is not so, for if we measure up by a rod, both ends of the interval are necessarily measured simultaneously, and $dt = 0$. The interval element then reduces to a line element given by

$$ds^2 = -\frac{dr^2}{1 - \dfrac{2m}{r}} - r^2 d\theta^2$$

and this cannot be changed into the form $dx_1^2 + dx_2^2$

At distances remote from matter the expression (11), as we have seen, reduces to that of the restricted theory. Under such conditions all the g's are unity, but they are not all of the same sign.* This characterizes what is called a semi-Euclidian continuum. If, however, measurements are made simultaneously, which can be effected by any observer in his own system, $dt = 0$, and he may regard his space as strictly Euclidian.

Some further interesting conclusions may be drawn from the relation (11). Let us take an interval ds measured in the direction of r only, so that $d\theta = 0$ and $dt = 0$. We have then—neglecting the negative sign—

$$ds^2 = \frac{dr^2}{1 - 2m/r}$$

or,

$$ds = \frac{dr}{\sqrt{1 - 2m/r}}.$$

Again, take an equal interval and measure it perpendicularly to r, so that $dr = 0$ and $dt = 0$. We then get

$$ds = rd\theta.$$

Since $dt = 0$ in both cases, ds represents an actual length as measured by a rod. Now dr and $rd\theta$ are what these lengths should measure up to if they are to obey the relation

$$ds^2 = dr^2 + r^2d\theta^2,$$

which is the condition for Euclidian space. But when measured along the radius vector, ds—that is to say the length of the rod—has to be multiplied by

* See previous footnote.

$\sqrt{1 - 2m/r}$ before it will fit into a Euclidian space. But $\sqrt{1 - 2m/r}$ is less than unity, and thus the rod contracts when placed radially. Transverse measurements require no change.*

If $r = 2m$, $1 - 2m/r = 0$, and the length of the measuring rod vanishes. Thus, as we approach an attracting particle there comes a time when the length of the measuring rod vanishes and no advance is made, however many times we apply it.† This need not be taken as representing an actual physical happening. It is only the way our equations have of telling us that matter is impenetrable. Another effect of a similar kind concerning time will be considered in the next chapter.‡

We are now in a position to consider more closely the amount of indebtedness of the general theory to the restricted theory. In Chapter XII all we carried forward with us was a sense of dissatisfaction at the limitation to unaccelerated systems, and it was stated that the general theory was logically independent of the restricted theory. This is strictly true, but it is also true that the restricted theory has furnished several valuable data without which the development of the general theory would have been practically impossible. It is inconceivable that the six equations embodying the general law of gravitation could have been stated

* Eddington, "Report," pp. 27 and 47.

† Eddington, "Space, Time, and Gravitation," p. 98.

‡ The reader will of course not confound these changes in lengths and times due to a gravitational field with those considered in connexion with the restricted theory, which occur in a non-gravitational field and are due to movement.

without reference to the restricted theory which led to the concept of the four-dimensional continuum. Nor would the particular solution which led to the expression (10) for the line element have suggested itself had it not been clear that this solution must resolve itself into the form given in Chapter XI.

The idea, referred to in Chapter XI, of taking the velocity of light as unity, has also led to considerable simplification, though this cannot be made so evident here as in a more detailed work, but one curious consequence of this convention needs remark. It brings out the fact that progress through the four-dimensional space-time continuum is very much more rapid in the time direction than in the space directions. That is to say, unless a particle is moving with a velocity comparable with that of light, the rate of change of its time co-ordinate with respect to any of its space co-ordinates is very great. For all ordinary objects, the "world-lines" are very nearly what we should call straight if we were speaking of progress through space only, and they are nearly parallel to the time axis. This will be clear from the fact that light travels at the rate of 300,000 kilometres per second. If, therefore, we call this velocity unit velocity, one second of time is the equivalent of 300,000 kilometres. Whence the time unit, one second, laid off along the time axis is 300,000 times the unit of length, or 1 kilometre, laid off any of the length axes.

It is impossible to give any accurate picture of what is happening in a gravitational field, according to Einstein's view, but it still seems possible to get some general idea by the aid of analogies, though all analogies

are necessarily rough or even grotesque. The new theory concentrates attention upon the courses which objects pursue, the older theory upon forces which are supposed to influence bodies so as to make them follow these courses.* According to the Newtonian view, in the absence of force all bodies have a natural path in space, namely, a straight line described with uniform velocity. If, however, any other body be present it exercises a pull on the first body, drawing it out of its natural path if the two bodies are not in the line of the path, but accelerating its motion, in any case. A corresponding pull is exerted by the first body on the second. If the new point of view which Einstein invites us to adopt, presents a difficulty, it is useful to remember that the Newtonian view presented no less difficulty to philosophers in his day. Their great objection was that it involved action at a distance, attributing to bodies a power to act where they are not. It seemed incredible that the sun acted across intervening space and pulled a planet out of the straight path which it would otherwise follow. Only the clearest evidence that this theory actually did give an explanation of the planetary motions, and presented a picture of what, in fact, went on in the solar system, surpassing by far in adequacy and accuracy any theory previously advanced, induced philosophers as a body to accept such action as possible. We have now grown so accustomed to it that nothing else seems natural to us.

Einstein does not deny the influence of matter, but he gives us a different picture. It is as though space

* Eddington, "Space, Time, and Gravitation," pp. 95, 96; *cf.* Chapter XIII (5).

were filled with some medium or substance—matter we must not call it—but some medium which acts as a guide to bodies passing through it. A solitary body is guided in a uniform straight course, or allowed to follow such a course; it is immaterial which way we put it. But now introduce into the medium another body. Forthwith there is set up a twist, or strain, or curvature in the medium throughout space, intense near the body in proportion to its massiveness, and fading away and eventually disappearing in remoter parts, but everywhere continuous without gaps or sudden changes, excepting in the places actually occupied by the bodies. The result of this action, which we figure as a change of structure of the medium, is to guide the first body into a curved path and to produce all the effects which we have grouped under the term acceleration. Of course, the first body distorts the medium in a similar way and sets up a corresponding disturbance in the motion of the second. This twist or curvature corresponds to what we have called the curvature of space. We notice that action at a distance is eliminated. The second body affects the medium in its immediate neighbourhood, and this portion affects those in contact with it, and so on, just as a disturbance set up in water by a moving body is propagated outwards and affects other bodies in the same mass of water.

The analogy just given is most imperfect, more especially because it is stated in terms of the ordinary three dimensions only. We have to suppose that all that has been described takes place in the four-dimensional space-time continuum, where pictorial diagrams fail and where the symbolical representations of mathe-

matics are all we have to depend upon. But by suppressing one of the three spatial dimensions, some sort of a picture may be made even of this state of affairs. Consider a bundle of straight rubber tubes, and suppose progress in the direction of their length to represent duration in time, movement in any other direction being displacement in space, as usual. A small particle projected down one of the tubes will thus appear to be growing old, but will appear to be fixed spatially. Now let a massive body be projected down a tube somewhere in the middle of the bundle, and imagine that the effect is to twist the bundle so that the tubes present the appearance of the strands of a rope. The tube containing the particle will thus be twisted into a helix like a screw, and the particle will be constrained to follow its course. The projection of this course on any cross section of the bundle of tubes will be an oval or round curve, and if for "particle" we read "planet" and "sun" for "massive body," we have a picture of a planet and its orbit. The tubes represent the world-lines of the bodies moving within them. In view of what has just been said in connexion with the unit value of the velocity of light, the helices which the tubes form will be so elongated axially as to be almost straight.

Summary.—The parallel between geometry and gravitation suggests that the g's which specify the one also specify the other. The embodiment in mathematical terms is a set of six equations, which constitute the law of gravitation. By means of these the g's are determined in a form consistent with the re-

stricted theory of relativity. Their values show that space is non-Euclidian in a gravitational field. In the absence of a gravitational field it is semi-Euclidian. In a gravitational field a measuring rod contracts when placed radially. Matter is mathematically expressed as a discontinuity. The general theory is logically independent of the restricted theory, but is stated so that the restricted theory is a particular or limiting case. World-lines are usually nearly "straight". The Newtonian view concentrates attention upon forces acting at a distance causing bodies to pursue certain courses, while the new view concentrates upon the courses themselves.

CHAPTER XIX

THE CRUCIAL PHENOMENA

THREE deductions which can be submitted to experimental tests have been made from the results of the last chapter. They are known as the "Crucial Phenomena," because they stand in relation to Einstein's theory as the necessary experimental complement which is required, as in the case of every hypothesis, before it can find acceptance. These phenomena relate to the following :—

(1) The motion of the apse of the planet Mercury.

(2) The bending of light rays by the sun.

(3) The displacement of lines in the Solar spectrum.

1. The Orbit of Mercury.

According to Newton's law, if the solar system consisted of two bodies only, the sun and a planet, the planet would describe round the sun an ellipse unvarying in shape, size, and position, and having the sun in one of its foci. The effect of the mutual attractions of the planets on one another, however, is to disturb the orbits in various ways. Amongst other things, each orbit rotates slowly, pivoted on the sun, while the planet revolves in it.

Thus, if a planet be pictured as a bead sliding on an

oval wire while the wire itself keeps turning round in the same direction as that in which the bead slides, an idea of the actual path of a planet in space relatively to the sun may be obtained. Or, still better, cut out an ellipse from a piece of card and fasten it down on a table by a pin through the focus *S*. Now take a pencil, and while rotating the card slowly and evenly about *S*, trace round the circumference of the card in the same direction, but more quickly. A pattern such as that shown in Fig. 36 will be obtained. The apses, which

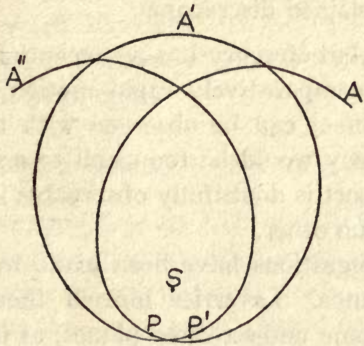

FIG. 36.

is the name given to the pointed ends of the orbit, *A*, *A′*, *A″*, etc., and *P*, *P′*, etc., where the planet is at its greatest and least distance from the sun, thus advance—that is, they turn round in the same direction as the revolution of the planet. Each apse advances every year through an angle such as *PSP′*. The French astronomer Leverrier, in the survey of the solar system, in the course of which he was led to the discovery of the planet Neptune, calculated amongst other things the amount of advance of the apses of the planets

according to Newton's law, and he found that the calculated amount agreed with the observed amount as nearly as could be expected in every case, excepting in that of Mercury. The apse of Mercury was observed to advance every year by considerably more than the calculated amount. Of course, this is comparatively speaking, for the actual amounts are exceedingly small. The figures are in seconds of arc per century :—

Observed advance . . .	574″
Calculated advance . . .	532″
Unexplained discrepancy . .	42″

Were it not that Mercury has an exceptionally pointed orbit and a comparatively rapid motion, so that the cumulative effect can be observed with relative ease, this discrepancy would be too small to notice. Something of the sort is doubtfully observable in the case of Mars, but in no other.

Various suggestions have been made to account for this discordance. Leverrier himself thought that it was due to some undiscovered planet, as in the case of Uranus, but this time inside the disturbed orbit.* But no such planet has ever been found, and all other explanations have similarly failed until the time of Einstein.

The actual work of finding an orbit is very similar whether we take Newton's or Einstein's law. In the latter case it is somewhat more complicated. But in principle the two points of view differ very materially. In the Newtonian case we suppose a planet, or particle as we may call it, to be started moving with a velocity

* Eddington, "Space, Time, and Gravitation," p. 124.

given in direction and magnitude at a stated place, and we find its path when some central body, the sun say, pulls on it with a force which varies according to Newton's law. If we adopt Einstein's way of looking at the matter we ask ourselves how the particle will move if projected with a given velocity as before, but otherwise moving freely, in so far as the curvature of space or the distortion due to the sun of the medium filling space will allow. The result of accepting the values of the forces deduced from the g's determined in the last chapter is to add a small quantity to Newton's statement, which accounts for the extra advance of Mercury's apse. Newton's law, though very nearly true, is only approximate.

2. The Bending of Light Rays

It was shown in Chapters XIII and XIV that straight lines in non-gravitational fields were distorted into curves in artificial gravitational fields, and the same thing happens in natural gravitational fields. A ray of light is straight *in vacuo* in the absence of gravitation, and it may therefore be expected to become curved in the presence of matter.

It has long been held by philosophers that light has mass. If this mass were subject to Newtonian gravitation, a ray of light passing near a heavy body such as the sun would follow a definite orbit and would be deflected. Einstein's theory, however, apart from the question of the mass of light, shows that the course of a ray depends definitely upon the geometry of the space through which it moves, and predicts an amount

of deflexion double of that which is to be expected from Newton's law. The only practicable test is to observe the apparent displacement of the fixed stars when light rays from them pass near the sun on their way to the earth. A star at T (Fig. 37) sends out rays in straight lines in all directions. One of these, TAE, strikes the earth E and renders the star visible; other

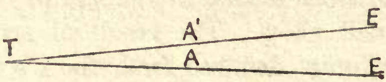

FIG. 37.

rays such as $TA'E'$ miss the earth. Now interpose the sun S near the paths of the rays. The effect is to bend the ray TA (Fig. 38) towards the sun into the direction AE_1, so that it now misses the earth. The ray which reaches the earth is the ray TA', which is now bent in the direction $A'E$. Thus the star is now seen at T' on EA' produced, instead of on ET as before, and the

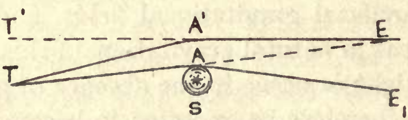

FIG. 38.

effect of the sun has been to displace its apparent position outwards. According to Einstein, this displacement should be about double the amount predicted by the older theory—1·74 seconds of arc as against ·87 of a second. These angles are very small (one second of arc is the angular diameter of a halfpenny over three miles away), and observations are possible only during a total eclipse, stars near the sun being lost in the sun's

rays and therefore invisible at any other time. The effect was observed during the eclipse of the sun of May 29, 1919. Photographs taken of the region near the sun were compared with photographs of the same region when the sun was out of the way, and the difference showed the effect to the satisfaction of the astronomers, though many of the photographs were spoilt by cloud. Several attempts have been made to explain the results of observation by the assumption of a dense refracting atmosphere near the sun, but the deflexion has resisted all explanations other than that upon which Einstein based his prediction.

3. Displacement of Lines in the Solar Spectrum

According to the electron theory of matter atoms are very highly complicated structures composed of minute bodies charged with negative electricity called electrons, revolving round a central nucleus like planets round the sun. Their periods of revolution show remarkable constancy for the same substance, and their motion produces all the effects of electricity in violent oscillation. They behave, in fact, like the oscillators in wireless telegraphy, and send out electro-magnetic waves in all directions. In wireless telegraphy these waves are large, but the waves set up by the electrons in an atom are excessively minute. When the atom is hot the disturbances which constitute the waves become intense, and those which are of the proper period become capable of affecting the eye with the sensation of light. The period remains constant for the same substance whatever the temperature, and consequently the electrons behave like remarkably

regular and efficient clocks. Now one of the things which Einstein showed was that the rates of clocks depend upon the value of g_4, and they lose time.* The vibrations of all the electrons are slowed down, and it can further be shown that the more rapid vibrations are retarded more in proportion than the slower ones. The result is that the violet rays, which correspond to the more rapid vibrations, are affected to a greater extent than the slower ones, which produce the colour red, and thus the spectrum becomes crowded up towards the red when the source of the light is in a strong gravitational field. It was, therefore, expected that light rays issuing from any particular substance in the sun would be displaced towards the red, compared with those issuing from the same substance on earth. The observations are extraordinarily difficult, and the evidence of the existence of the effect is conflicting.

The failure to detect this prediction of Einstein may be due either (1) to the extreme difficulty of the observations; (2) to the existence of some other effect which masks the displacement, but which is as yet undiscovered; or (3) to the failure of Einstein's theory. This third alternative does not necessarily involve the untruth of the whole theory, but only its inapplicability to the phenomena of radiation, of which light is a particular case. The matter at stake is in fact the applicability of Einstein's theory to quantum phenomena. An outline of the quantum theory was given in Chapter II (4), and it is regarded as of the very highest importance by physicists. It is, in fact, of

* Eddington, "Report," p. 56.

the same order of importance as the atomic theory of matter, and it has found general acceptance. If, therefore, it were once definitely shown that Einstein's theory failed in its applicability to this class of phenomena, its generality would be very seriously impaired. So far there does not appear to be any conclusive evidence one way or the other.

The position therefore stands that Einstein has explained an outstanding difficulty in the Newtonian theory; it has predicted an unlooked-for effect on light and explained it, while the truth of a second prediction is in doubt. The balance of experimental evidence is therefore so far in favour of the truth of the theory, but, in addition to this evidence, there are other considerations which are not strictly experimental, but which, taken together, tend in the same direction. These, however, will find a more fitting place in the next chapter.

Summary.—The experimental tests of the theory relate to (1) the orbit of Mercury; (2) the bending of light rays in a gravitational field; (3) the crowding up of spectral lines towards the red end. The first two have been verified, but not the third The importance of the third lies in its relation to the quantum theory.

CHAPTER XX

THE APPLICATION OF THE GENERAL PRINCIPLE

THE principle of equivalence has thus led to the practical identification of geometry and gravitation, but this result, important and even impressive though it is, must not be allowed to obscure the main issue. It is, after all, only a means to an end, the end being the application of the General Principle of Relativity to the statement of physical laws. In order to complete the subject it has still to be shown how the gravitation theory enables these laws to be stated in identical forms, no matter what systems of reference are used.

This work is still incomplete. Should the theory of relativity find general acceptance, it will occupy physicists for many years to come. Those who can understand the somewhat advanced mathematics which are required, will find an indication of the initial steps in Professor Eddington's report to the Physical Society on the Relativity Theory of Gravitation, to which frequent reference has been made in this book. The whole work will mean the complete re-writing of mathematical physics in the new terms. The mathematical limits of the present book prevent any attempt in this direction. In dealing with the restricted principle we

had to stop short at the application, and the same obstacle, exaggerated on account of the still more advanced mathematics required, hinders us here. The following observations must therefore be of a very general character.

When phenomena are referred to the four-dimensional space-time continuum, it is found that the general facts of physics can be expressed in terms resembling the invariant expression for the interval element in that they do not alter their form, whatever reference system is used. Alteration of the reference system, as we have seen, introduces artificial forces, which are indistinguishable from gravitational forces. The change of co-ordinates and the consequent change in the forces cancel one another out, so to speak, and preserve the form of the mathematical expressions which the change of co-ordinates might otherwise be expected to modify. It is as though the alteration in point of view brought into action an automatic governor, and switched in some agency which maintained the balance. The supreme importance of gravitation is thus manifest. Instead of being, as heretofore, a thing apart amongst natural agencies, it assumes, as it were, a controlling place. It is the counterpart of the geometry in terms of which all physical phenomena must be stated, and it corresponds with the circumstances of every observer so that he can make his statements in forms identical with those of any other.

A very remarkable consequence follows from this. Since all reference systems are equivalent, they may change from time to time without any corresponding change occurring in statements of law. The changes

themselves produce the necessary balance. It matters not whether, according to the standards of A, B's system is rigid or not. All that can happen is that B's geometry will differ from that of A, and a corresponding difference will arise between their gravitational fields and produce automatic compensation. If B assumes his system to be rigid he necessarily assumes that A's standards are not, for it was by these standards that A judged the rigidity. Thus neither observer need attribute rigidity to the other's system or standards, though he assumes it for his own. The same considerations apply to the regularity of clocks. Everybody uses his own local measures, and all express the general results in identical form.

This relieves us from the necessity of defining rigidity. Just as it is impossible to define position or motion without reference to some object, so we cannot define rigidity without reference to some body assumed to be rigid. This body again requires comparison with a third in order to test its rigidity, and so on without limit. We now see that no such definition is necessary. We define any length as so many multiples or submultiples of a standard unit, and this unit is the distance between two marks on a metal bar under specified physical conditions. This is our standard length, and no further trouble need be taken to ascertain whether it is rigid in the absolute sense or not. So also for clocks. We may take anything we like as the standard of time—a rotation of the earth, the time of vibration of a sodium atom or any other convenient unit.

The preceding chapters may have created the uncomfortable impression on the reader's mind that he has

been led into the somewhat mystifying region called a four-dimensional space-time continuum and left there to extricate himself as best he can. The reflexion that this region can be represented adequately by mathematical symbols will probably not afford much relief. This feeling is not likely to oppress those who are accustomed to mathematical symbols, and who know, for example, the very practical physical results which follow from investigations involving imaginary expressions, such, for instance, as the square root of -1. The difficulty in removing this impression, if it exists, lies in the very limited amount of mathematics which the writer is allowing himself. It seems to require a much smaller amount of mathematics to get into a four-dimensional continuum than to get out of it again. The reader may therefore be reminded that he has not been led into this region. What has been done is to point out that he and everything else were in it already, and always have been there. We may repeat the oft-quoted words of Minkowski : " The views of space and time, which I have set forth, have their foundation in experimental physics. Therein is their strength. Their tendency is revolutionary. From henceforth space in itself and time in itself sink into mere shadows, and only a kind of union of the two preserves an independent existence." *

As a matter of fact, these investigations result in relations between space and time which are no more essentially mysterious than those to which everyone is accustomed in civil life, though the method by which

* Eddington, "Space, Time, and Gravitation," p. 30.

the results were attained involves this unfamiliar combination of the two.*

It has already been seen in Chapter XI that the physical history of every object is its world-line. The whole of physical nature in the mathematical diagram is a mass of these world-lines existing in the four-dimensional continuum like strings in a piece of jelly, and sometimes intersecting one another. The intersections of the world-lines of observers with other world-lines mark phenomena. The essential order of these intersections is not disturbed by distortion any more than the order of the intersections of the strings in the jelly, though it might so appear to particular individuals. It is this order which matters. It is independent of any individual point of view, which is the same thing as saying that the imposition of any particular reference system, or system of co-ordinates, makes no difference to it. The physical laws of nature which are stated in invariant fashion concern these world-lines, and the fact that they are independent of any particular co-ordinate system is therefore only what we should expect.

The effect upon many, perhaps most, minds of the study of the application of the General Principle to Physics, is to create a strong bias in favour of the truth of the theory. This, of course, can hardly be said to be evidence, but when one sees familiar and well-established results coming out of it as well as new ones, an impression of coherence and unity is created which appeals to the artistic instincts if not to the strictly

* *E.g.*, the results in Chapter XIX.

scientific ones. Besides what is called its heuristic power—the power of finding things out—the theory seems to promise a unification of physical knowledge on a scale hitherto deemed impossible. It is the dream of some enthusiasts that it may be the means of unifying all knowledge, and that it may one day lead to the expression of all activities by a single equation. The writer has his doubts. He finds it difficult to fancy a sermon resolving itself into a blackboard demonstration with a differential equation as the text.

Summary.—The Gravitation Theory is a step in the application of the General Principle to the statement of Physical Laws. This application is in process of being worked out. Gravitation is now linked up with other physical agencies. The equivalence of reference systems renders definition of rigidity of length standards or regularity of time standards unnecessary. The results of inquiries conducted in terms of the four-dimensional continuum are expressed in the ordinary terms of three-dimensional space and one-dimensional time. Physical nature is made up of world-lines. The order of the intersections of world-lines is the important fact in nature, and this order is unaffected by any choice of reference systems. The heuristic value of the theory, and its power of unifying knowledge, create an impression of its truth.

CHAPTER XXI

GENERAL SUMMARY AND CONCLUSION

WE have now come to the end of the subject as defined by the title of the book and introduced the Theory of Relativity. There is nothing left but to summarize, and to add some final observations.

After defining relativity as the theory of the statement of general physical laws so as to express them in identical forms, in spite of differences in the points of view of observers, the vague idea of a point of view was crystallized into the more precise concept of a reference system, a kind of framework fitted out with clocks, which is essential to the numerical statement of all phenomena, and which is, or may be, peculiar to every observer. Those comprehensive statements of fact called general physical laws were next considered. It appeared that these statements, since they all relate to measurement, must be expressed in mathematical terms, and that the subject matter of relativity relates to such expressions. It appeared further that an essential feature of these laws must be identity of form for different observers, since statements holding good for individuals, or small groups of observers only, cannot be called general, and are of no value as a means of putting facts on record for the benefit of others.

Physics, so far from being a coherent body of knowledge, would be but a Babel. But it was seen that by making use of unaccelerated rectangular systems, or Galilean systems as they were called, all mechanical laws could be stated in identical forms. Though space and time measurements might be made in units peculiar to each system, the general expressions comprehending the facts to which the measurements related, reduced to identical forms for all such systems, so that a general law applied everywhere though interpreted according to the several measurements of individuals. It was seen, however, that for such statements to be possible it was necessary to assume that measured lengths and times were not altered by relative movement between the systems; that, for example, a yard measure on A's system meant to B the same as a yard on his own, and similarly with units of time, so that two observers could attribute the same length to the same object, or the same interval of time, notwithstanding their relative movement. If these assumptions were made, peculiarities of individual systems, such as relative velocity, dropped out of account and mechanical laws showed no preference for one system over another. These assumptions, in fact, made it possible to act upon the principle that all Galilean systems are equally suitable for the statement of general mechanical laws, and this statement was called the Mechanical Principle of Relativity. When, however, it was sought to extend the application of this principle from mechanical laws to electro-magnetic laws, electro-magnetic laws appeared to have a preference for a reference system at rest in the medium in which electro-magnetic agencies

operate. If the laws were stated in terms of a Galilean system moving with reference to this medium, the velocity of the system entered into the statements and so deprived them of generality. But it further appeared that if the suppositions regarding identity of measurements of lengths and times were abandoned and replaced by certain others, according to which the lengths of objects measured in the direction of motion, and measured times, did not appear the same to two observers in relative motion, electro-magnetic laws preserved their identity of form no matter to what Galilean system they were referred. These new suppositions were, however, recognized as being more or less empirical, though supported by electro-magnetic considerations. Einstein showed that these new suppositions could be derived from the remarkable fact that the velocity of light is the same relative to every observer. This fact makes the velocity of light unique amongst all other velocities, but it follows directly from the two postulates :—

(1) That no observer can detect his own motion through the medium which transmits light.

(2) That the velocity of light *in vacuo* is independent of that of its source.

The two new suppositions were thus put upon a basis independent of any electro-magnetic considerations. With their aid it was possible to extend the Mechanical Principle of Relativity and to act upon the principle that all Galilean reference systems are equally suitable for the statement of general *physical* laws. This is called the Restricted Principle of Relativity, because its operation is limited to Galilean reference systems.

The new suppositions led to some remarkable conclusions respecting estimates of lengths and times, velocities, masses, the simultaneousness of events, and so forth, and were embodied in a set of equations called the Lorentz transformation, by which physical events and geometrical statements in any Galilean system could be related to those in any other. It was further seen that the new suppositions necessitated modification of mechanical laws from their original form.

It was seen that the invariant expression for a line element contained as many terms as there were dimensions of space. The Lorentz transformation was applied to the three-dimensional expression, and the result showed that when this transformation is used, an invariant expression must contain four terms, a time term being a necessary addition. The concept of objects as four-dimensional, and existing in a four-dimensional continuum, is thus the necessary consequence of the use of the Lorentz transformation. Though no actual picture can be formed corresponding to this concept, it can be represented adequately by mathematical symbolism.

No logical reason being assignable for restricting the statements of physical laws to unaccelerated systems, the conditions were examined under which it might be possible to express them in terms of Gaussian systems. It was seen that the forces which were necessarily introduced form an obstacle which could, however, be overcome by the adoption of the Principle of Equivalence. An examination of the principle of equivalence brought out a parallel between gravitational fields of force and their geometry, which led to Einstein's supposition that

the quantities which enter as multipliers into the expression for the interval element also specify the gravitational forces in any region. This supposition can be formulated definitely in a set of six equations which constitute Einstein's hypothetical law of gravitation, and a solution of these can be obtained, advantage being taken of the fact that the expression for the interval element must reduce to the restricted theory form. This solution shows that when matter is present space is non-Euclidian. Einstein proposed three deductions from his hypothesis as crucial tests, and two of these have been confirmed. The automatic introduction of gravitational forces consequent upon change of co-ordinates enables the form of expressions of physical laws, when stated in terms of the four-dimensional space-time continuum, to be preserved. The application of the principle that all Gaussian four-dimensional co-ordinate systems are equivalent for the statement of general physical laws is thus made possible. This principle is called the General Principle of Relativity.

The Æther

In the previous pages no special pains have been taken to draw any clear distinction between space and the medium called the æther, which philosophers have supposed to pervade space, and which serves as the vehicle for the transmission of light and other electromagnetic radiations, and generally as the seat of actions not ascribable to matter. No distinction seemed necessary, for as long as it was realized that light was conveyed and that curvature or twisting existed, questions as to what conveyed the light, or

what it was that was twisted or curved, did not seem to be of immediate relevance. This omission, however, was a matter of convenience. Something further will now be said on the subject.

It is very widely held that facts are against the existence of the æther, and that Einstein's theory dispenses with the need of it. As far as the writer understands the matter, it is argued that because one set of experiences require a fixed æther, while another set fail to detect movement through it, it can neither be moving nor fixed, and that therefore it cannot exist. It seems, however, to the writer that those who, on the one hand, are lamenting the death of the æther, and on the other are executing war dances over its corpse, are over hasty. If by æther is meant something which has properties such as mass, impenetrability, rigidity, or elasticity, which are usually associated with matter, then, indeed, the æther is dead. But if there is no medium of any kind, and nothing in space, we are compelled, so it seems to the writer, to attribute to empty vacuity the properties of transmitting light, and of assuming geometrical structure, which is very like a contradiction in terms, if not actually so. It is impossible to accept the supposition that nothing can do anything, even transmit light waves.

Now, though it may be difficult to conceive of anything which has none of the ordinary properties of matter, but is yet capable of the activities which have just been named, there is no actual contradiction involved. It is the softer horn of the dilemma. There is, of course, the third alternative that æther and space are one and the same, but this seems to require us to

believe that there must of necessity be something in the interval between two bodies which, as it were, props them apart, and which, if removed, would cause or allow the interval to collapse. There is no evidence for any such supposition. There is nothing to show that nature abhors a vacuum to the extent of making two bodies coalesce. If, therefore, we are, as it would seem, driven to accept the fact that the universe of experience is filled with some entity which cannot be called matter, it is merely a question of words whether we call this entity æther or not.

The fact, as it appears to the writer, is that so far from Einstein having destroyed the æther or rendered it superfluous, he has discovered in it the capacity of assuming some sort of geometrical structure in time-space. It is nothing new to think of the æther as subject to strain and thus capable of exhibiting geometrical properties, but this fresh capacity seems to be something of quite a different order. It may very well be that this is the beginning of the discovery of a series of properties which use and time may eventually weld into one concept, and that the æther, so far from being dead, is in process of being born.

Action at a Distance

The immediate predecessor of Newton's theory was the Cartesian theory of Vortices. According to this theory, space is filled with a subtle medium or æther which is in a continual state of whirl, producing vortices which entangle bodies such as the planets and thus cause them to revolve. The theory of vortices broke down under analysis, but it was held to possess an

advantage over Newton's theory in that the cause of the motion was present where the effect occurred, while, according to Newton's theory, the cause resided in a distant body. Newton's theory, therefore, required a body to act where it was not, and this was held to be inconceivable.

As a way out of the difficulty, it has been suggested that since the sun manifestly acts upon the planets, and if action at a distance is impossible, then the sun must, in a sense, be present where the planets are. This suggestion, fantastic though it is, at least serves to show how strongly the idea of efficiency or adequacy, as part of the concept of cause, has impressed men's minds. It has been urged, if not as an argument in favour of Einstein's theory, at least as one of its advantages, that it dispenses with the idea of action at a distance. It does not assume, as the theory of Descartes seems to assume, that motion is caused or maintained by æther, but it implies that motion is in some sense determined or guided by an agency present where the body is, and acting directly upon it by contact. It does not give an explanation, any more than Newton's theory, of the agency which started the body moving, but it is held that it gives an intelligible picture of its subsequent movement, in which particular Newton's theory is thought to fail.

While holding that efficiency is a proper and necessary part of the concept of cause in the philosophical sense of the term, the writer is unable to see that the theories of either Einstein or Descartes offer any advantage in this respect over Newton's theory. He is quite unable to understand how or why contact or collision between

bodies modifies their motion. It is a known fact that it does do so, but so does distant action—at least, so we have become accustomed to think since the time of Newton, but the mechanism which produces this effect is just as mysterious in the one case as in the other. It is all the more mysterious where, as in the present case, the action occurs between a material body and a subtle substance like the æther, which is held on other grounds not to be impenetrable. Even when two material bodies collide, and it might be held that impenetrability obliges one or other to give way, it still remains to explain impenetrability. This seems, to the writer, to be a most obscure property. If, as we now believe, matter is made up of atoms separated by great distances, each atom being composed of electrons separated by distances which are enormous relatively to their size, it is an extraordinary fact that bodies cannot pass through one another without action upon either. Electric fields may be invoked to explain it, but this is only action at a distance over again.

The Limited Universe

The curvature of space, or of the æther, leads to the conclusion that any region, if sufficiently extended, may eventually bend round into itself, and thus that the universe of experience may be limited. Indeed, calculations have been made as to its dimensions, the amount of matter in it, and so forth. This does not necessarily mean that the universe is bounded. For, consider two-dimensional beings on the surface of a sphere. Their universe is limited to the surface, but

they can wander about it freely without encountering a boundary.

The use of the term universe in this connexion is somewhat unfortunate. It is open to the construction that nothing can possibly exist outside a limited region. All that can be meant is that there are geometrical limits to man's experience, and this, if true, is a highly important addition to knowledge. If there is anything bigger than this " universe," or if there is more matter anywhere, it cannot come within our knowledge, just as no velocity greater than that of light is measurable. The statement may be nothing more than the mathematical expression of the imprisonment of mankind in the present state of existence.

Philosophy

The separation of mathematics and physics from metaphysics, explained in Chapter II, is a matter of method only, and must not be held to imply that metaphysics is thereby ruled out of account as a serious subject of inquiry. Some such procedure had to be adopted if any progress were to be made in knowledge, owing to the failure of metaphysics to reach positive conclusions. It is an interesting matter for speculation whether the ancients or their successors would have thought it worth while to devote so much energy to philosophic speculation if they had grasped the possibilities of the method of hypothesis backed by experiment. It is, in the writer's belief, fortunate that they did not. They might have been diverted from philosophical inquiries to such an extent as to allow it to be

forgotten that there was anything in such concepts as being, cause, space or time, other than those parts separated out for treatment by the mathematicians and physicists. It is not wrong to make this separation; it is a matter of necessity, and no error is imported by it into mathematics or physics. No error, for example, arises in physics from ignoring efficiency as part of the concept of cause, and defining cause as a necessary antecedent. It may be a great deal more, but it certainly is that, and whatever else it may be, does not concern the physicist as such. But though no error is entailed upon physics by this limitation, very serious error might be entailed upon human thought by forgetting such matters as that cause might imply very much besides invariable antecedence. Philosophical speculation, barren though it has been in positive results, has played an important part in keeping to the front the belief that there may be other things in the universe besides the material. Einstein's theory points in the same direction. The remarkable feature about it is that starting from a purely experimental basis, it compels us to accept the supersensual as a fact. If the theory is true this conclusion seems inevitable.

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